

## Chapter 15 Difference Equations and the Z-Transforms

### 15-1 The First-order Linear Difference Equation $y_{k+1} = a_k y_k + b_k$ , $a_k \neq 0$

**Solution:** 
$$\frac{1}{\prod_{i=0}^k a_i} y_{k+1} - \frac{a_k}{\prod_{i=0}^k a_i} y_k = \frac{b_k}{\prod_{i=0}^k a_i} \Rightarrow \frac{y_{k+1}}{\prod_{i=0}^k a_i} - \frac{y_k}{\prod_{i=0}^{k-1} a_i} = \frac{b_k}{\prod_{i=0}^k a_i}$$

Let  $A_k = \frac{y_k}{\prod_{i=0}^{k-1} a_i} \Rightarrow A_{k+1} - A_k = \frac{b_k}{\prod_{i=0}^k a_i}$ ,  $\sum_{j=0}^{k-1} [A_{j+1} - A_j] = A_k - A_0 = \sum_{j=0}^{k-1} \frac{b_j}{\prod_{s=0}^j a_s}$

$$\Rightarrow y_k = \prod_{i=0}^{k-1} a_i \cdot \left[ A_0 + \sum_{j=0}^{k-1} \frac{b_j}{\prod_{s=0}^j a_s} \right]$$

**Eg. Solve  $y_{k+1} + 2^k y_k = 1$ .**

(Sol.)  $y_{k+1} = -2^k y_k + 1$ ,  $a_k = -2^k$ ,  $a_0 a_1 a_2 \cdots a_k = (-1)^{k+1} \cdot 2^{\frac{k(k+1)}{2}}$

$$\frac{y_{k+1}}{(-1)^{k+1} \cdot 2^{\frac{k(k+1)}{2}}} = \frac{y_k}{(-1)^k \cdot 2^{\frac{(k-1)k}{2}}} + \frac{1}{(-1)^{k+1} \cdot 2^{\frac{k(k+1)}{2}}}$$

$$A_k = \frac{y_k}{(-1)^k \cdot 2^{(k-1)k/2}}, \quad A_{k+1} = \frac{y_{k+1}}{(-1)^{k+1} \cdot 2^{k(k+1)/2}}$$

$$A_{k+1} - A_k = (-1)^{-1-k} \cdot 2^{\frac{k(k+1)}{2}}, \quad A_k - A_0 = \sum_{i=0}^{k-1} (-1)^{-1-i} \cdot 2^{\frac{i(i+1)}{2}}$$

$$A_k = A_0 + \sum_{i=0}^{k-1} (-1)^{-1-i} \cdot 2^{\frac{i(i+1)}{2}}, \quad y_k = (-1)^k \cdot 2^{\frac{k(k-1)}{2}} \cdot \left[ A_0 + \sum_{i=0}^{k-1} (-1)^{-1-i} \cdot 2^{\frac{i(i+1)}{2}} \right]$$

**Eg. Solve  $y_{k+1} = \frac{k}{k+1} y_k + 4$ .**

(Sol.)  $a_k = \frac{k}{k+1} \neq 0$  for  $k \neq 0$ ,  $a_1 a_2 \cdots a_k = \frac{1}{k+1}$

$$\frac{y_{k+1}}{1/(k+1)} - \frac{k}{k+1} \cdot \frac{y_k}{1/(k+1)} = \frac{4}{1/(k+1)}, \quad (k+1)y_{k+1} - ky_k = 4(k+1)$$

Let  $A_k = ky_k$ ,  $A_{k+1} = (k+1)y_{k+1}$ ,  $A_{k+1} - A_k = 4(k+1)$

$$\sum_{j=1}^{k-1} [A_{j+1} - A_j] = A_k - A_1 = 4 \cdot \frac{(k-1)(k+2)}{2}$$

$$A_k = A_1 + 2(k-1)(k+2) = ky_k \Rightarrow y_k = \frac{1}{k} [A_1 + 2(k-1)(k+2)]$$

**Eg. Solve  $y_{k+1}=ky_k$ .**

$$\begin{aligned} \text{(Sol.) } a_k = k, \quad a_1 a_2 \cdots a_k = k!, \quad \frac{y_{k+1}}{k!} &= \frac{y_k}{(k-1)!}. \text{ Let } A_k = y_k / (k-1)!, \quad A_{k+1} = y_{k+1} / k! \\ A_{k+1} = A_k &\Rightarrow A_k = A_1 \\ \Rightarrow y_k &= A_1 (k-1)! \end{aligned}$$

**Eg. Solve  $y_{k+1}=y_k+k^2-k$ .**

$$\begin{aligned} \text{(Sol.) } y_{k+1} - y_k = k^2 - k, \quad y_k - y_0 &= \sum_{n=0}^{k-1} (n^2 - n) = \sum_{n=0}^{k-1} n^2 - \sum_{n=0}^{k-1} n \\ &= \frac{k(k-1)(2k-1)}{6} - \frac{(k-1)k}{2} = \frac{k(k-1)(k-2)}{3} \\ \therefore y_k &= y_0 + \frac{k(k-1)(k-2)}{3} \end{aligned}$$

**Notations:**  $Dy_k = y_{k+1} - y_k, \quad D^2 y_k = D[Dy_k] = D[y_{k+1} - y_k] = y_{k+2} - 2y_{k+1} + y_k, \dots,$   
 $D^n y_k = y_{k+n} - C_1^n y_{k+n-1} + C_2^n y_{k+n-2} - \dots + y_k$

## 15-2 The Second-order Constant-coefficient Difference Equations

$y_{k+2} + ay_{k+1} + by_k = f_k$

**Homogeneous form:  $y_{k+2} + ay_{k+1} + by_k = 0$**

Suppose  $y_k = r^k, \quad r^{k+2} + ar^{k+1} + br^k = 0, \quad r^2 + ar + b = 0$

**Case 1**  $r = r_1, \quad r_2 \in R, \quad r_1 \neq r_2 \Rightarrow y_k = c_1 r_1^k + c_2 r_2^k$

**Eg. Solve  $y_{k+2} + 2y_{k+1} - 3y_k = 0$ .**

(Sol.)  $r^2 + 2r - 3 = 0, \quad r = 1, -3 \Rightarrow y_k = c_1 (1)^k + c_2 (-3)^k = c_1 + c_2 (-3)^k$

**Case 2**  $r = r_1 = r_2 \Rightarrow y_k = c_1 r_1^k + c_2 k r_1^k$

**Eg. Solve  $y_{k+2} - 8y_{k+1} + 16y_k = 0$ .**

(Sol.)  $r^2 - 8r + 16 = 0, \quad r = 4, 4 \Rightarrow y_k = c_1 4^k + c_2 k 4^k$

**Case 3**  $r = \alpha \pm i\beta = \rho e^{\pm i\theta} \Rightarrow y_k = c_1 (\alpha + i\beta)^k + c_2 (\alpha - i\beta)^k$   
 $= \rho^k [d_1 \cos(k\theta) + d_2 \sin(k\theta)]$

**Eg. Solve  $y_{k+2} - 2y_{k+1} + 2y_k = 0$ .**

(Sol.)  $r^2 - 2r + 2 = 0, \quad r = 1 \pm i = \sqrt{2} e^{\pm j\frac{\pi}{4}} \Rightarrow y_k = (\sqrt{2})^k \left[ c_1 \cos\left(\frac{k\pi}{4}\right) + c_2 \sin\left(\frac{k\pi}{4}\right) \right]$

**Eg. Given the following recurrence equation:  $r_n=r_{n-1}+2r_{n-2}$  with  $r_0=1, r_1=1$ . (a) Please answer the values of  $r_2$ , and  $r_{10}$ . (b) Please find the general expression for  $r_n$  in terms of  $n$ . [2004 台大電研]**

(Sol.) (b) Set  $n-2=m, n=m+2$ , and then reset  $m=n$ . We have  $r_{n+2}-r_{n+1}-2r_n=0$ .

$$r^2 - r - 2 = 0, r=2, -1 \Rightarrow r_n = c_1 2^n + c_2 (-1)^n$$

$$r_0=1, r_1=1 \Rightarrow c_1=2/3, c_2=1/3 \Rightarrow r_n = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n$$

$$(a) r_2 = \frac{2}{3} \cdot 2^2 + \frac{1}{3} \cdot (-1)^2 = 3 \quad \text{and} \quad r_{10} = \frac{2}{3} \cdot 2^{10} + \frac{1}{3} \cdot (-1)^{10} = 683$$

**Method of undetermined coefficients:**

**A particular solution  $y_k^{(p)}$  is chosen by these rules:**

1.  $f_k$  is a polynomial,  $y_k^{(p)}$  may be a polynomial.
2.  $f_k$  is a combination of sinusoidal functions, so is  $y_k^{(p)}$ .
3.  $f_k$  is an exponential function, so is  $y_k^{(p)}$ .

**Eg. Solve  $y_{k+2}-7y_{k+1}+12y_k=-16+12k$ .**

$$(Sol.) y_{k+2} - 7y_{k+1} + 12y_k = 0 \Rightarrow y_k^{(h)} = c_1 3^k + c_2 4^k$$

$$\text{Let } y_k^{(p)} = A + Bk, \quad y_{k+1} = A + B(k+1), \quad y_{k+2} = A + B(k+2)$$

$$y_{k+2} - 7y_{k+1} + 12y_k = A + B(k+2) - 7[A + B(k+1)] + 12[A + Bk] = -16 + 12k$$

$$6A - 5B + 6Bk = -16 + 12k$$

$$\Rightarrow \begin{cases} 6A - 5B = -16 \\ 6B = 12 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 2 \end{cases} \Rightarrow y_k^{(p)} = -1 + 2k$$

$$\therefore y_k = c_1 3^k + c_2 4^k - 1 + 2k$$

**Eg. Solve  $y_{k+2}+8y_{k+1}+12y_k=e^k$ .**

$$(Sol.) y_k^{(h)} = c_1 (-2)^k + c_2 (-6)^k$$

$$\text{Let } y_k^{(p)} = Ae^k, \quad y_{k+1} = Ae^{k+1} = Ae \cdot e^k, \quad y_{k+2} = Ae^{k+2} = Ae^2 \cdot e^k$$

$$y_{k+2}^{(p)} + 8y_{k+1}^{(p)} + 12y_k^{(p)} = A(e^2 + 8e + 12)e^k = e^k$$

$$\Rightarrow A = \frac{1}{e^2 + 8e + 12} \Rightarrow y_k^{(p)} = \frac{e^k}{e^2 + 8e + 12}$$

$$\therefore y_k = c_1 (-2)^k + c_2 (-6)^k + \frac{e^k}{e^2 + 8e + 12}$$

**Eg. Solve  $y_{k+2}-4y_k=\sin(k)$ .**

(Sol.)  $y_k^{(h)} = c_1 2^k + c_2 (-2)^k$ . Let  $y_k^{(p)} = A \sin(k) + B \cos(k)$

$$\begin{aligned} \Rightarrow A \sin(k+2) + B \cos(k+2) - 4A \sin(k) - 4B \cos(k) \\ = [A \cos(2) - B \sin(2) - 4A] \sin(k) + [A \sin(2) + B \cos(2) - 4B] \cos(k) = \sin(k) \end{aligned}$$

$$\Rightarrow \begin{cases} A \cos(2) - B \sin(2) - 4A = 1 \\ A \sin(2) + B \cos(2) - 4B = 0 \end{cases} \Rightarrow \begin{aligned} A &= \frac{\cos(2) - 4}{17 - 8 \cos(2)} \\ B &= \frac{-\sin(2)}{17 - 8 \cos(2)} \end{aligned}$$

$$\therefore y_k = c_1 2^k + c_2 (-2)^k + \frac{1}{17 - 8 \cos(2)} [(\cos(2) - 4) \sin(k) - \sin(2) \cos(k)]$$

**Eg. Solve  $y_{k+2}+3y_{k+1}-4y_k=20k$ .**

(Sol.)  $y_{k+2} + 3y_{k+1} - 4y_k = 0 \Rightarrow y_k^{(h)} = c_1 \cdot 1^k + c_2 (-4)^k = c_1 + c_2 (-4)^k$

1.  $y_k^{(p)} = A + Bk$ ,  $A$  and  $B$  are independent of  $k$ .

$$\Rightarrow A + B(k+2) + 3A + 3B(k+1) - 4A - 4Bk = 20k$$

$$\Rightarrow B = 4k \text{ is not a constant.}$$

2.  $y_k^{(p)} = A + Bk + Ck^2$

$$\begin{aligned} \Rightarrow A + B(k+2) + C(k+2)^2 + 3A + 3B(k+1) + 3C(k+1)^2 - 4A - 4Bk - 4Ck \\ = (5B + 7C) + 10Ck = 20k \end{aligned}$$

$$\Rightarrow \begin{aligned} 5B + 7C = 0 &\Rightarrow B = -\frac{14}{5} \\ 10C = 20 &\Rightarrow C = 2 \end{aligned} \Rightarrow y_k^{(p)} = -\frac{14}{5}k + 2k^2$$

$$\therefore y_k = c_1 + c_2 (-4)^k - \frac{14}{5}k + 2k^2$$

**Eg. Solve  $y_{k+2}-4y_{k+1}+4y_k=3 \cdot 2^k+5 \cdot 4^k$ . [1991 清大化工所]**

(Ans.)  $y_k = c_1 2^k + c_2 k 2^k + \frac{3}{8} k^2 2^k + \frac{5}{4} 4^k$

**Variation-of-parameter method to find the particular solutions:**

For a homogeneous second-order linear difference equation, if  $y_k^{(1)}$  and  $y_k^{(2)}$  are solution of it, then  $c_1 y_k^{(1)} + c_2 y_k^{(2)}$  is also a solution for arbitrary  $c_1$  and  $c_2$ .

**Wronskian determinant:**  $W_k = \begin{vmatrix} y_k^{(1)} & y_k^{(2)} \\ y_{k+1}^{(1)} & y_{k+1}^{(2)} \end{vmatrix} = y_k^{(1)} y_{k+1}^{(2)} - y_k^{(2)} y_{k+1}^{(1)}$

Determine the particular solution of  $y_{k+2} + a_k y_{k+1} + b_k y_k = f_k$ .

Suppose  $y_k^{(h)} = c_1 y_k^{(1)} + c_2 y_k^{(2)}$ , then let  $y_k^{(p)} = A_k y_k^{(1)} + B_k y_k^{(2)}$

$$\begin{aligned} y_{k+1}^{(p)} &= A_{k+1} y_{k+1}^{(1)} + B_{k+1} y_{k+1}^{(2)} \\ &= A_k y_{k+1}^{(1)} + B_k y_{k+1}^{(2)} + [y_{k+1}^{(1)} (A_{k+1} - A_k) + y_{k+1}^{(2)} (B_{k+1} - B_k)] \\ &= A_k y_{k+1}^{(1)} + B_k y_{k+1}^{(2)} + y_{k+1}^{(1)} DA_k + y_{k+1}^{(2)} DB_k \end{aligned}$$

Impose the condition  $y_{k+1}^{(1)} DA_k + y_{k+1}^{(2)} DB_k = 0 \dots\dots(1)$

$$\begin{aligned} \Rightarrow y_{k+1}^{(p)} &= A_k y_{k+1}^{(1)} + B_k y_{k+1}^{(2)} \\ y_{k+2}^{(p)} &= A_{k+1} y_{k+2}^{(1)} + B_{k+1} y_{k+2}^{(2)} \\ &= y_{k+2}^{(1)} [A_{k+1} - A_k] + y_{k+2}^{(2)} [B_{k+1} - B_k] + A_k y_{k+2}^{(1)} + B_k y_{k+2}^{(2)} \\ &= y_{k+2}^{(1)} DA_k + y_{k+2}^{(2)} DB_k + A_k y_{k+2}^{(1)} + B_k y_{k+2}^{(2)} \\ \Rightarrow y_{k+2}^{(p)} + a_k y_{k+1}^{(p)} + b_k y_k^{(p)} &= y_{k+2}^{(1)} DA_k + y_{k+2}^{(2)} DB_k + A_k [y_{k+2}^{(1)} + a_k y_{k+1}^{(1)} + b_k y_k^{(1)}] + B_k [y_{k+2}^{(2)} + a_k y_{k+1}^{(2)} + b_k y_k^{(2)}] \\ &= f_k \\ \Rightarrow y_{k+2}^{(1)} DA_k + y_{k+2}^{(2)} DB_k &= f_k \dots\dots(2) \end{aligned}$$

by (1) & (2):  $\begin{cases} DA_k = A_{k+1} - A_k = -\frac{f_k y_{k+1}^{(2)}}{W_{k+1}} \\ DB_k = B_{k+1} - B_k = \frac{f_k y_{k+1}^{(1)}}{W_{k+1}} \end{cases}$

$$\Rightarrow \begin{cases} A_k - A_0 = \sum_{j=0}^{k-1} \frac{-f_j y_{j+1}^{(2)}}{W_{j+1}} \\ B_k - B_0 = \sum_{j=0}^{k-1} \frac{f_j y_{j+1}^{(1)}}{W_{j+1}} \end{cases} \Rightarrow \begin{cases} A_k = \sum_{j=0}^{k-1} \frac{-f_j y_{j+1}^{(2)}}{W_{j+1}} \\ B_k = \sum_{j=0}^{k-1} \frac{f_j y_{j+1}^{(1)}}{W_{j+1}} \end{cases} \text{ if we choose } A_0=B_0=0.$$

$$\Rightarrow y_k = c_1 y_k^{(1)} + c_2 y_k^{(2)} + \left( \sum_{j=0}^{k-1} \frac{-f_j y_{j+1}^{(2)}}{W_{j+1}} \right) \cdot y_k^{(1)} + \left( \sum_{j=0}^{k-1} \frac{f_j y_{j+1}^{(1)}}{W_{j+1}} \right) \cdot y_k^{(2)}$$

**Eg. Solve  $y_{n+2}+8y_{n+1}+7y_n=ne^n$ .**

(Sol.)  $y_n^{(h)} = c_1(-1)^n + c_2(-7)^n$

$$W_{n+1} = \begin{vmatrix} (-1)^{n+1} & (-7)^{n+1} \\ (-1)^{n+2} & (-7)^{n+2} \end{vmatrix} = -6 \cdot 7^{n+1}$$

$$-f_j y_{j+1}^{(2)} = -je^j \cdot (-7)^{j+1}, \quad f_j y_{j+1}^{(1)} = je^j \cdot (-1)^{j+1}$$

$$-f_j y_{j+1}^{(2)} / W_{j+1} = \frac{-je^j (-7)^{j+1}}{-6 \cdot 7^{j+1}} = \frac{je^j (-7)^{j+1}}{6 \cdot 7^{j+1}}$$

$$f_j y_{j+1}^{(1)} / W_{j+1} = \frac{je^j (-1)^{j+1}}{-6 \cdot 7^{j+1}} = \frac{-je^j (-1)^{j+1}}{6 \cdot 7^{j+1}}$$

$$\therefore y_n = c_1(-1)^n + c_2(-7)^n + \left[ \sum_{j=0}^{n-1} \frac{je^j (-7)^{j+1}}{6 \cdot 7^{j+1}} \right] \cdot (-1)^n + \left[ \sum_{j=0}^{n-1} \frac{-je^j (-1)^{j+1}}{6 \cdot 7^{j+1}} \right] \cdot (-7)^n$$

### 15-3 Cauchy-Euler Difference Equation $k(k+1)D^2y_k+akDy_k+by_k=0$

**Solution:** Let  $y_k = \frac{\Gamma(k+r)}{\Gamma(k)} = k(k+1)\dots(k+r-1)$ , where  $\Gamma(n+1) = n!$  and  $n$  is

an integer.

$$Dy_k = \frac{\Gamma(k+r+1)}{\Gamma(k+1)} - \frac{\Gamma(k+r)}{\Gamma(k)} = \frac{(k+r)\Gamma(k+r)}{k\Gamma(k)} - \frac{\Gamma(k+r)}{\Gamma(k)} = \frac{r}{k} \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$D^2y_k = Dy_{k+1} - Dy_k = y_{k+2} - 2y_{k+1} + y_k$$

$$= \frac{\Gamma(k+r+2)}{\Gamma(k+2)} - 2 \cdot \frac{\Gamma(k+r+1)}{\Gamma(k+1)} + \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$= \left[ \frac{(k+r+1)(k+r)}{(k+1)k} - 2 \cdot \frac{(k+r)}{k} + 1 \right] \cdot \frac{\Gamma(k+r)}{\Gamma(k)} = \frac{r(r-1)}{k(k+1)} \cdot \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$\Rightarrow k(k+1)D^2y_k + akDy_k + by_k = (r^2 - r + ar + b) \frac{\Gamma(k+r)}{\Gamma(k)} = 0$$

$$\Rightarrow r^2 + (a-1)r + b = 0 \Rightarrow \text{Solve } r.$$

**Eg. Solve  $k(k+1)D^2y_k-5kDy_k+8y_k=0$ .**

(Sol.)  $a=-5, b=8 \Rightarrow r^2 - 6r + 8 = 0 \Rightarrow r = 2, 4$

$$\frac{\Gamma(k+2)}{\Gamma(k)} = k(k+1), \quad \frac{\Gamma(k+4)}{\Gamma(k)} = k(k+1)(k+2)(k+3),$$

$$\therefore y_k = c_1 k(k+1) + c_2 k(k+1)(k+2)(k+3)$$

## 15-4 The Z-transform and Its Application

**Discrete Laplace transform:**  $L[f^*(t)] = L[f(nT)] = \sum_{n=0}^{\infty} f(nT)e^{-nTs}$

Let  $Z=e^{Ts}$ , and define Z-transform as  $F(Z)=z[f(t)]=z[f^*(t)]=\sum_{n=0}^{\infty} f(nT)Z^{-n}$

**Properties of Z-transform  $z[f(t)]=F(Z)$ :**

1.  $z[c_1f_1(t)+c_2f_2(t)]=c_1F_1(Z)+c_2F_2(Z)$

2.  $z[f(t+T)]=Z[F(Z)-f(0^+)]$ ,  $z[f(t+2T)]=Z^2F(Z)-Z^2f(0^+)-Zf(T)$

$$z[f(t+nT)]=Z^n \left[ F(Z) - \sum_{k=0}^{n-1} f(kT)Z^{-k} \right] = Z^n F(Z) - Z^n f(0) - Z^{n-1} f(T) - Z^{n-2} f(2T) - Z^{n-3} f(3T) - \dots - Zf((n-1)T)$$

$$\begin{aligned} \text{(Proof)} \quad z[f(t+nT)] &= \sum_{m=0}^{\infty} f(mT+nT)Z^{-m} = Z^n \sum_{m=0}^{\infty} f(mT+nT)Z^{-(m+n)} \\ &= Z^n \sum_{k=n}^{\infty} f(kT)Z^{-k} = Z^n \left[ \sum_{k=0}^{\infty} f(kT)Z^{-k} - \sum_{k=0}^{n-1} f(kT)Z^{-k} \right] = Z^n \left[ F(Z) - \sum_{k=0}^{n-1} f(kT) \cdot Z^{-k} \right] \end{aligned}$$

3.  $z[f(t-T)]=Z^{-1}F(Z)$ ,  $z[f(t-nT)u(t-nT)]=Z^{-n}F(Z)$

$$\begin{aligned} \text{(Proof)} \quad z[f(t-nT)u(t-nT)] &= \sum_{m=0}^{\infty} f(mT-nT)u(mT-nT)Z^{-m} \\ &= Z^{-n} \sum_{m=0}^{\infty} f(mT-nT)u(mT-nT)Z^{-(m-n)} = Z^{-n} \sum_{k=0}^{\infty} f(kT)Z^{-k} = Z^{-n}F(Z) \end{aligned}$$

4.  $z[e^{-at}f(t)]=F(e^{aT}Z)$

$$\text{(Proof)} \quad z[e^{-at}f(t)] = \sum_{n=0}^{\infty} e^{-naT} f(nT)Z^{-n} = \sum_{n=0}^{\infty} f(nT)(e^{+aT}Z)^{-n} = F(e^{aT}Z)$$

5.  $z \left[ \sum_{k=0}^n f(kT) \right] = \frac{Z}{Z-1} F(Z)$  for  $|Z|>1$

6.  $z[tf(t)] = -TZ \frac{d}{dZ} F(Z)$ ,  $z[t^k f(t)] = -TZ \frac{d}{dZ} \{z[t^{k-1} f(t)]\}$

(Proof)

$$z[tf(t)] = \sum_{n=0}^{\infty} nTf(nT)Z^{-n} = -TZ \sum_{n=0}^{\infty} f(nT)(-nZ^{-n-1}) = -TZ \frac{d}{dZ} \left[ \sum_{n=0}^{\infty} f(nT) \cdot Z^{-n} \right] = -TZ \frac{d}{dZ} F(Z)$$

**7.  $f(0^+) = \lim_{Z \rightarrow \infty} F(Z)$ : the initial-value theorem**

$\lim_{n \rightarrow \infty} f(nT) = \lim_{Z \rightarrow 1} (Z-1)F(Z)$ : the final-value theorem

(Proof)  $F(Z) = \sum_{n=0}^{\infty} f(nT)Z^{-n} = f(0T) + \frac{f(T)}{Z} + \frac{f(2T)}{Z^2} + \dots, \therefore \lim_{Z \rightarrow \infty} F(Z) = f(0)$

$$z[f(t+T) - f(t)] = \lim_{n \rightarrow \infty} \sum_{k=0}^n [f((k+1)T) - f(kT)]Z^{-k} = ZF(Z) - Zf(0) - F(Z)$$

$$\lim_{Z \rightarrow 1} \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(k+1)T - f(kT)]Z^{-k} = \lim_{n \rightarrow \infty} \{ [f(T) - f(0)] + [f(2T) - f(T)] + \dots + [f(n+1)T - f(nT)] \}$$

$$= \lim_{n \rightarrow \infty} [-f(0) + f(n+1)T] \Rightarrow \lim_{n \rightarrow \infty} f(nT) = \lim_{n \rightarrow \infty} f((n+1)T) = \lim_{Z \rightarrow 1} (Z-1)F(Z)$$

8.  $z\left[\frac{\partial}{\partial a} f(t, a)\right] = \frac{\partial}{\partial a} F(Z, a), \quad z\left[\int_{a_1}^{a_2} f(t, a) da\right] = \int_{a_1}^{a_2} F(Z, a) da$

**Eg. Determine  $z[1]$ ,  $z[t]$ ,  $z[a^t]$ ,  $z[e^{-at}]$ ,  $z[\sin(at)]$ , and  $z[\cos(at)]$ .**

(Sol.)  $f(t) = 1 \Rightarrow f(nT) = 1, \forall n$

$$z[1] = \sum_{n=0}^{\infty} f(nT)Z^{-n} = \sum_{n=0}^{\infty} Z^{-n} = \frac{1}{1 - \frac{1}{Z}} = \frac{Z}{Z-1}$$

$$\begin{aligned} z[t] &= z[nT] = \sum_{n=0}^{\infty} nTZ^{-n} = (-TZ) \sum_{n=0}^{\infty} (-nZ^{-n-1}) = -TZ \sum_{n=0}^{\infty} \frac{d}{dZ} (Z^{-n}) = -TZ \frac{d}{dZ} \left( \sum_{n=0}^{\infty} Z^{-n} \right) \\ &= -TZ \frac{d}{dZ} \left( \frac{Z}{Z-1} \right) = -TZ \cdot \frac{Z-1-Z}{(Z-1)^2} = \frac{TZ}{(Z-1)^2} \end{aligned}$$

$$z[a^t] = z[a^{nT}] = \sum_{n=0}^{\infty} a^{nT} \cdot Z^{-n} = \sum_{n=0}^{\infty} \left( \frac{a^T}{Z} \right)^n = \frac{1}{1 - \frac{a^T}{Z}} = \frac{Z}{Z - a^T}$$

$$z[e^{-at}] = z[e^{-anT}] = \sum_{n=0}^{\infty} e^{-naT} \cdot Z^{-n} = \sum_{n=0}^{\infty} (e^{aT} Z)^{-n} = \frac{1}{1 - (e^{aT} Z)^{-1}} = \frac{Z}{Z - e^{-aT}}$$

$$\begin{aligned} z[e^{+jat}] &= z[e^{+janT}] = z[\cos(anT)] + jz[\sin(anT)] \\ &= \frac{Z}{Z - e^{+jaT}} = \frac{Z(Z - e^{-jaT})}{(Z - e^{+jaT})(Z - e^{-jaT})} \\ &= \frac{Z^2 - Z \cos(aT) + jZ \sin(aT)}{Z^2 - (e^{jaT} + e^{-jaT})Z + 1} = \frac{Z^2 - Z \cos(aT) + jZ \sin(aT)}{Z^2 - 2Z \cos(aT) + 1} \end{aligned}$$

$$\therefore z[\sin(at)] = z[\sin(naT)] = \frac{Z \sin(aT)}{Z^2 - 2Z \cos(aT) + 1}$$

$$z[\cos(at)] = z[\cos(naT)] = \frac{Z^2 - Z \cos(aT)}{Z^2 - 2Z \cos(aT) + 1}$$



**Table of the Z-transform pairs:  $T \neq 1$**

$f(nT)$	$F(Z)$
$\delta(nT)$	1
1 or $u(t)$	$\frac{Z}{Z-1}$
$nT$	$\frac{TZ}{(Z-1)^2}$
$e^{-naT}$	$\frac{Z}{Z-e^{-aT}}$
$a^{nT}$	$\frac{Z}{Z-a^T}$
$\sin(naT)$	$\frac{Z \sin(aT)}{Z^2 - 2Z \cos(aT) + 1}$
$\cos(naT)$	$\frac{Z^2 - Z \cos(aT)}{Z^2 - 2Z \cos(aT) + 1}$

**Table of the Z-transform pairs:  $T=1$**

$f(n)$	$F(Z)$
$\delta(n)$	1
1	$\frac{Z}{Z-1}$
$n$	$\frac{Z}{(Z-1)^2}$
$e^{-na}$	$\frac{Z}{Z-e^{-a}}$
$a^n$	$\frac{Z}{Z-a}$

**Inverse Z-transform:** If  $F(Z)/Z = \frac{A_1}{Z-p_1} + \frac{A_2}{Z-p_2} + \dots + \frac{A_k}{Z-p_k}$ , and then we have  
 $f(n) = A_1 p_1^n + A_2 p_2^n + \dots + A_k p_k^n$

**Eg. For  $F(Z) = \frac{Z^2}{(Z-1)(Z-0.5)}$ ,  $f(n) = ?$**

(Sol.)  $\frac{F(Z)}{Z} = \frac{Z}{(Z-1)(Z-0.5)} = \frac{2}{Z-1} - \frac{1}{Z-0.5} \Rightarrow f(n) = 2 \cdot 1^n - \left(\frac{1}{2}\right)^n = 2 - \left(\frac{1}{2}\right)^n$

**Eg. Solve  $a_{n+1}-0.5a_n=1, a_0=1$ .**

(Sol.) Let  $a_n=f(n), z[a_n]=F(Z), z[a_{n+1}]=ZF(Z)-Zf(0)=ZF(Z)-Z, a_0=f(0)=1$

$$ZF(Z) - Z - \frac{1}{2}F(Z) = \frac{Z}{Z-1}, \quad \left(Z - \frac{1}{2}\right)F(Z) = \frac{Z^2}{Z-1}$$

$$\frac{F(Z)}{Z} = \frac{Z}{(Z-1)\left(Z - \frac{1}{2}\right)} = \frac{2}{Z-1} - \frac{1}{Z - \frac{1}{2}} \Rightarrow f(n) = 2 - \left(\frac{1}{2}\right)^n = a_n$$

**Eg. Solve  $a_n-0.5a_{n-1}=1, a_0=2$ .**

(Sol.) Let  $m=n-1, n=m+1 \Rightarrow a_{m+1} - \frac{1}{2}a_m = 1 \Rightarrow a_{n+1} - \frac{1}{2}a_n = 1,$

$$z(a_{n+1}) - \frac{1}{2}z(a_n) = z(1) \Rightarrow ZF(Z) - 2Z - \frac{1}{2}F(Z) = \frac{Z}{Z-1},$$

$$\left(Z - \frac{1}{2}\right)F(Z) = \frac{Z + 2Z^2 - 2Z}{Z-1} = \frac{2Z^2 - Z}{Z-1}, \quad \frac{F(Z)}{Z} = \frac{2}{Z-1} \Rightarrow f(n) = a_n = 2$$

**Eg. Solve  $r_n=r_{n-1}+2r_{n-2}$  with  $r_0=1, r_1=1$ . [2004 台大電研]**

(Sol.) Let  $m=n-2, n=m+2 \Rightarrow r_{m+2}-r_{m+1}-2r_m=0 \Rightarrow r_{n+2}-r_{n+1}-2r_n=0,$   
 $z(r_{n+2})-z(r_{n+1})-2z(r_n)=0.$

By  $z(r_n)=R(Z), z(r_{n+1})=Z[R(Z)-f(0^+)], z(r_{n+2})=Z^2R(Z)-Z^2f(0^+)-Zf(T),$  and define  $r_0=f(0^+),$   
 $r_1=f(T) \Rightarrow [Z^2R(Z)-Z^2r_0-Zr_1]-[ZR(Z)-Zr_0]-2R(Z)=0$

$$\Rightarrow [Z^2-Z-2]R(Z)=Z^2, \quad \frac{R(Z)}{Z} = \frac{Z}{Z^2-Z-2} = \frac{\frac{2}{3}}{Z-2} + \frac{\frac{1}{3}}{Z+1}, \quad \therefore r_n = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n$$