Chapter 2 The Second-Order Ordinary Differential Equations

2-1 Introduction

Initial-value problem

Eg. $y''-2y=x^2-1$, $y(1)=3$, $y'(1)=-5$.

Boundary-value problem

Eg. $y''+y=0$, $y(0)=y(\pi)=0$.

Theorem Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of $y''+P(x)y'+Q(x)y=0$, then $y(x)=c_1y_1(x)+c_2y_2(x)$ is its general solution.

2-2 The 2-order Linear Constant-coefficient Ordinary Differential Equation

$y''+Ay'+By=F(x)$

Homogeneous equation: $y''+Ay'+By=0$

Let $y=e^{rx}$, $r^2+Ar+B=0 \Rightarrow r=\dfrac{-A\pm\sqrt{A^2-4B}}{2}$

Case 1 $A^2-4B>0 \Rightarrow r=\dfrac{-A\pm\sqrt{A^2-4B}}{2}=r_1,\ r_2,\ r_1\neq r_2,\ \therefore\ \ y(x)=c_1e^{r_1x}+c_2e^{r_2x}$

Eg. Solve $y''+3y'+2y=0$.

(Sol.) $r^2+3r+2=0$, $r=-1, -2$, $\therefore\ y(x)=c_1e^{-x}+c_2e^{-2x}$

Case 2 $A^2-4B=0 \Rightarrow r=\dfrac{-A}{2},\ \ y_1(x)=e^{-\frac{Ax}{2}},\ \ y_2=u(x)e^{-\frac{Ax}{2}}$

$$y_2''(x)+Ay_2'(x)+By_2(x)=0 \Rightarrow u(x)=c_1x+c_2$$

Choose $\quad c_1=1,\ c_2=0,\ y_2(x)=xe^{-\frac{Ax}{2}},\ \therefore\ \ y(x)=c_1e^{-\frac{Ax}{2}}+c_2xe^{-\frac{Ax}{2}}$

Eg. Solve $y''+4y'+4y=0$.

(Sol.) $r^2+4r+4=0$, $r=-2, -2$, $\therefore\ y(x)=c_1e^{-2x}+c_2xe^{-2x}$

Case 3 $A^2-4B<0 \Rightarrow r=\dfrac{-A\pm i\sqrt{4B-A^2}}{2}=p\pm iq$

$$y(x)=c_1e^{(p+iq)x}+c_2e^{(p-iq)x}=d_1e^{px}\cdot\cos(qx)+d_2e^{px}\cdot\sin(qx)$$

Eg. Solve $y''+9y=0$.

(Sol.) $r^2+9=0$, $r=\pm i3$, $\therefore\ y(x)=c_1\cos(3x)+c_2\sin(3x)$

Eg. Solve $y''+2y'+26y=0$.

(Sol.) $r^2+2r+26=0$, $r=-1\pm5i$, $\therefore\ \ y(x)=c_1e^{-x}\cos(5x)+c_2e^{-x}\sin(5x)$

Eg. Two students solve $y''+ay'+by=0$, $y(0)=A$ and $y'(0)=B$. Using wrong constants for b and B, one student obtain the solution $y_A=e^{-2x}(\cos 3x+2\sin 3x)$. Using wrong constants for a and A, one student obtain the solution $y_B=-3e^x+2e^{3x}$. Find the correct constants for a, b, A, and B and solve the initial value problem. [2001 台大電研]

(Sol.) $y_A=e^{-2x}(\cos 3x+2\sin 3x) \Rightarrow r=-2\pm i3 \Rightarrow r^2+4r+13=0=r^2+ar+b$,

$\because$ b is wrong but a is correct, $\therefore$ $a=4$

$\because$ $y(0)=A$ is correct, $\therefore$ $y_A(0)=e^0(\cos 0+2\sin 0)=1=A$

$y_B=-3e^x+2e^{3x} \Rightarrow r=1, 3 \Rightarrow r^2-4r+3=0=r^2+ar+b$, $\because$ a is wrong but b is correct, $\therefore$ $b=3$

$y_B'(x)=-3e^x+6e^{3x}$, $\because$ $y'(0)=B$ is correct, $\therefore$ $y_B'(0)=-3+6=3=B$

The correct $r^2+ar+b=r^2+4r+3=0 \Rightarrow r=-1, -3 \Rightarrow$ The correct $y(x)=ce^{-x}+de^{-3x}$

$y(0)=A=1 \Rightarrow c+d=1$ and $y'(0)=B=3 \Rightarrow -c-3d=3$, we obtain $c=3$, $d=-2$

$$\Rightarrow y(x)=3e^{-x}-2e^{-3x}$$

Non-homogeneous equation: $y''+Ay'+By=F(x)$

1. Find the homogeneous solution y_h of $y''+Ay'+By=0$,
2. Find a particular solution y_p of $y''+Ay'+By=F(x)$, 3. General solution is y_h+y_p.

Eg. Solve $y''-4y=8x^2-2x$.

(Sol.) $r^2-4=0$, $r=2,-2$, $\therefore$ $y_h=c_1e^{-2x}+c_2e^{2x}$,

$y_p=ax^2+bx+c$, $y_p'=2ax+b$, $y_p''=2a$, $\therefore$ $y_p''-4y_p=-4ax^2-4bx+2a-4c=8x^2-2x$

$\Rightarrow a=-2, b=\dfrac{1}{2}, c=-1$, $\therefore$ $y(x)=c_1e^{-2x}+c_2e^{2x}-2x^2+\dfrac{1}{2}x-1$

Eg. Solve $y''+2y'-3y=4e^{2x}$.

(Sol.) $r^2+2r-3=0$, $r=1,-3$, $\therefore$ $y_h=c_1e^x+c_2e^{-3x}$.

Let $y_p=Ae^{2x}$, $y_p'=2Ae^{2x}$, $y_p''=4Ae^{2x}$, $\therefore$ $y_p''+2y_p'-3y_p=4e^{2x} \Rightarrow A=\dfrac{4}{5}$,

$\therefore$ $y=c_1e^x+c_2e^{-3x}+\dfrac{4}{5}e^{2x}$

Eg. Solve $y''+2y'-3y=4e^x$.

(Sol.) $r^2+2r-3=0$, $r=1,-3 \Rightarrow y_h=c_1e^x+c_2e^{-3x}$, $\therefore$ $y_p \neq Ae^x$,

Try $y_p=Axe^x$, $y_p'=Ae^x+Axe^x$, $y_p''=2Ae^x+Axe^x$,

$\therefore$ $y_p''+2y_p'-3y_p=4e^x \Rightarrow A=1$, $\therefore$ $y=c_1e^x+c_2e^{-3x}+xe^x$

Eg. Solve $y''+4y=\cos(x)$.

(Sol.) $r^2+4=0$, $r=\pm 2i$, $\therefore$ $y_h=c_1\cos(2x)+c_2\sin(2x)$.

Let $y_p=A\cos(x)+B\sin(x)$, $y_p'=-A\sin(x)+B\cos(x)$, $y_p''=-A\cos(x)-B\sin(x)$,

$y_p''+4y_p=3A\cos(x)+3B\sin(x)=\cos(x) \Rightarrow A=\dfrac{1}{3}$, $B=0$, $\therefore$ $y=c_1\cos(2x)+c_2\sin(2x)+\dfrac{1}{3}\cos(x)$

Eg. Solve $y''+4y=\cos(2x)$.

(Sol.) $r^2+4=0$, $r=\pm 2i$, $\therefore$ $y_h=c_1\cos(2x)+c_2\sin(2x)$ but $y_p\neq A\cos(2x)+B\sin(2x)$

Try $y_p=Ax\cos(2x)+Bx\sin(2x)$, $y_p'=A\cos(2x)-2Ax\sin(2x)+B\sin(2x)+2Bx\cos(2x)$,

$y_p''=-2A\sin(2x)-2A\sin(2x)-4Ax\cos(2x)+2B\cos(2x)+2B\cos(2x)-4Bx\sin(2x)$,

$y_p''+4y_p=-4A\sin(2x)+4B\cos(2x)=\cos(2x)\Rightarrow A=0,\ B=\frac{1}{4}$, $\therefore$ $y=c_1\cos(2x)+c_2\sin(2x)+\frac{x}{4}\sin(2x)$

Variation of parameters to find the particular solution y_p:

Let y_1 and y_2 be linearly independent solutions of $y''+Ay'+By=0$, then a particular solution y_p is $y_p=u(x)y_1(x)+v(x)y_2(x)$, and

$$y_p'=u'y_1+v'y_2+uy_1'+vy_2'=uy_1'+vy_2'$$

Impose the condition: $u'y_1+v'y_2=0\cdots\cdots(1)$

$$y_p''=u'y_1'+v'y_2'+uy_1''+vy_2''$$

$$\Rightarrow u'y_1'+v'y_2'+uy_1''+vy_2''+A\left(uy_1'+vy_2'\right)+B(uy_1+vy_2)=F(x)$$

$$\Rightarrow u\left[y_1''+Ay_1'+By_1\right]+v\left[y_2''+Ay_2'+by_2\right]+u'y_1'+v'y_2'=F(x)$$

$$u'y_1'+v'y_2'=F(x)\cdots\cdots(2)$$

$$(1),\ (2)\Rightarrow u'=\frac{-y_2F(x)}{y_1y_2'-y_2y_1'},\quad v'=\frac{y_1F(x)}{y_1y_2'-y_2y_1'}$$

Wronskian determinant: $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$

$$u(x)=\int\frac{-y_2F(x)}{W(y_1,y_2)}dx,\quad v(x)=\int\frac{y_1F(x)}{W(y_1,y_2)}dx\Rightarrow y_p=u(x)y_1(x)+v(x)y_2(x)$$

Eg. Find y_p of $y''-\frac{4}{x}y'+\frac{4}{x^2}y=x^2+1$. [1991 清大電研]

(Sol.) $y_1=x,\ y_2=x^4,\ W(y_1,y_2)=x\cdot 4x^3-x^4\cdot 1=3x^4$,

$$u(x)=\int\frac{-x^4\left(x^2+1\right)}{3x^4}dx=-\frac{x^3}{9}-\frac{x}{3},\quad v(x)=\int\frac{x\left(x^2+1\right)}{3x^4}dx=\frac{1}{3}\ell n|x|-\frac{1}{6x^2}$$

$$y_p=u(x)y_1+v(x)y_2=-\frac{x^4}{9}-\frac{x^2}{2}+\frac{x^4}{3}\ell n|x|$$

$\therefore$ $y(x)=y_h+y_p=cx+dx^4-\frac{x^4}{9}-\frac{x^2}{2}+\frac{x^4}{3}\ell n|x|=c_1x+c_2x^4-\frac{x^2}{2}+\frac{x^4}{3}\ell n|x|$

Eg. Solve *y*”+4*y*=tan(2*x*).

(Sol.) $y''+4y=0 \Rightarrow y_1=\cos(2x)$, $y_2=\sin(2x)$

$$W(y_1,y_2)=y_1y_2'-y_2y_1'=2\cos^2(2x)-\sin(2x)[-2\sin(2x)]=2$$

$$u(x)=\int\frac{-\sin(2x)\cdot\tan(2x)}{2}dx=\frac{1}{4}\sin(2x)-\frac{1}{4}\ell n\left|\tan\left(\frac{\pi}{4}+x\right)\right|$$

$$v(x)=\int\frac{\cos(2x)\cdot\tan(2x)}{2}dx=-\frac{1}{4}\cos(2x)$$

$$y_p=u(x)y_1(x)+v(x)y_2(x)$$

$$=\frac{1}{4}\sin(2x)\cos(2x)-\frac{\cos(2x)}{4}\cdot\ell n\left|\tan\left(\frac{\pi}{4}+x\right)\right|-\frac{1}{4}\cos(2x)\sin(2x)$$

$$=-\frac{1}{4}\cos(2x)\cdot\ell n\left|\tan\left(\frac{\pi}{4}+x\right)\right|$$

Eg. Solve 4*y*”+36*y*=csc(3*x*).

(Ans.) $y=a\cos(3x)+b\sin(3x)-\dfrac{x\cos(3x)}{12}+\dfrac{\sin(3x)\cdot\ln[|\sin(3x)|]}{36}$

Eg. Solve *y*”+4*y*=sec(2*x*). [中原電子所]

2-3 Euler Equation $x^2y''+Axy'+By=F(x)$

Solution: $z=ln(x)$, $y'=\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$,

$$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}\text{, and } x^2y''=-\frac{dy}{dz}+\frac{d^2y}{dz^2},\quad xy'=\frac{dy}{dz}$$

$$\Rightarrow \frac{d^2y}{dz^2}+(A-1)\frac{dy}{dz}+By=F(e^z)$$ is the second-order linear ODE.

Eg. Solve $x^2y''-5xy'+8y=2ln(x)+x^3$.

(Sol.) $z=ln(x)$, $y'=\frac{1}{x}\frac{dy}{dz}$, $y''=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$, $\frac{d^2y}{dz^2}-6\frac{dy}{dz}+8y=2z+e^{3z}$

$y_h=c_1e^{2z}+c_2e^{4z}=c_1x^2+c_2x^4$ and $y_p=az+b+ce^{3z}$, $y_p'=a+3ce^{3z}$, $y_p''=9ce^{3z}$,

$y_p''-6y_p'+8y_p=2z+e^{3z}\Rightarrow 8a=2$, $-6a+8b=0$, $-c=1\Rightarrow a=1/4$, $b=3/16$, $c=-1$,

$\therefore\quad y(x)=c_1x^2+c_2x^4+\frac{1}{4}\ell n(x)+\frac{3}{16}-x^3$

Eg. Solve $xy''+4y'=\frac{\ln x^3}{x}$. [2004 台大電研]

(Sol.) $x^2y''+4xy'=3ln(x)$, $z=ln(x)$, $y'=\frac{1}{x}\frac{dy}{dz}$, $y''=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$,

$\frac{d^2y}{dz^2}+3\frac{dy}{dz}=3z$, $y_h=c_1e^0+c_2e^{-3z}=c_1+c_2x^{-3}$ and $y_p=az^2+bz+c$, $y_p'=2az+b$, $y_p''=2a$,

$y_p''+3y_p'=3z\Rightarrow 6a=3$, $2a+3b=0\Rightarrow a=1/2$, $b=-1/3$, $\therefore\ y(x)=c_1+c_2x^{-3}+\frac{(\ln x)^2}{2}-\frac{\ln x}{3}$

Eg. Solve $x^2y''-2y=1/x$. [文化電機轉學考] (Ans.) $y(x)=cx^2+\frac{d}{x}-\frac{\ln(x)}{3x}$

Eg. Solve (a) $x^2y''-4xy'+4y=0$ and (b) $x^2y''+5xy'+4y=0$.

(Sol.) (a) Let $y=x^r$, $y'=rx^{r-1}$, $y''=r(r-1)x^{r-2}\Rightarrow x^2y''-4xy'+4y=x^r(r^2-5r+4)=0$

$\Rightarrow r=1, 4\Rightarrow y=c_1x+c_2x^4$.

(b) $y=x^r\Rightarrow r^2+4r+4=0$, $r=-2, -2\Rightarrow y_1=x^{-2}$, $y_2=x^{-2}ln(x)\Rightarrow y(x)=c_1x^{-2}+c_2x^{-2}\ell n(x)$

Another method: $z=ln(x)$, $y'=\frac{1}{x}\frac{dy}{dz}$, $y''=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$

$\Rightarrow\frac{d^2y}{dz^2}+4\frac{dy}{dz}+4y=0\Rightarrow\ y_1=e^{-2z}=x^{-2}$, $y_2=ze^{-2z}=x^{-2}ln(x)\Rightarrow y(x)=c_1x^{-2}+c_2x^{-2}\ell n(x)$

Eg. Solve (a) $x^2y''-xy'+y=ln(x)$ and (b) $x^2y''-4xy'+4y=x^4+x^2$. [交大電子所]

(Ans.) (a) $y(x)=c_1x+c_2x\ell n(x)+\ell n(x)+2$, (b) $y(x)=c_1x+c_2x^4+\frac{1}{3}x^4\ell nx-\frac{1}{2}x^2$

Eg. Solve $(x+2)^2y''-(x+2)y'+y=3x+4$. [2012 師大應用電子所]

(Sol.) $z=ln(x+2)$, $y'=\frac{1}{(x+2)}\frac{dy}{dz}$, $y''=\frac{-1}{(x+2)^2}\frac{dy}{dz}+\frac{1}{(x+2)^2}\cdot\frac{d^2y}{dz^2}$,

$\frac{d^2y}{dz^2}-2\frac{dy}{dz}+y=3e^z-2$, $y_1=e^z=x+2$, $y_2=ze^z=(x+2)\cdot ln(x+2)$, and $y_p=az^2e^z+b$,
$y_p'=2aze^z+az^2e^z$, $y_p''=2ae^z+4aze^z+az^2e^z \Rightarrow y_p''-2y_p'+y_p=(2ae^z+4aze^z+az^2e^z)-2(2aze^z+az^2e^z)+az^2e^z+b=2ae^z+b=3e^z-2 \Rightarrow a=3/2$, $b=-2 \Rightarrow y_p=3z^2e^z/2-2=3(x+2)^2\cdot ln(x+2)/2-2$,

$\therefore\ y(x)=c_1(x+2)+c_2(x+2)\cdot ln(x+2)+\frac{3}{2}(x+2)^2\cdot ln(x+2)-2$

Eg. Solve $(x-2)^2y''+4(x-2)y'+6y=0$. [文化電機轉學考]

(Sol.) Let $y=(x-2)^r$, $r^2+3r+6=0$, $r=\frac{-3\pm i\sqrt{15}}{2}$,

$$\therefore\ y(x)=c_1(x-2)^{-\frac{3}{2}}\cdot\cos\left[\frac{\sqrt{15}}{2}\ell n(x-2)\right]+c_2(x-2)^{-\frac{3}{2}}\cdot\sin\left[\frac{\sqrt{15}}{2}\ell n(x-2)\right]$$

2-4 Miscellaneous Problems

Eg. Solve $y'=(y+x)^3-1$. [2012台大電子所甲組]

(Sol.) Let $u=y+x$, $\frac{du}{dx}=y'+1=u^3$, $\frac{du}{u^3}=dx$, $\frac{-1}{2u^2}=x+c \Rightarrow \frac{-1}{2(y+x)^2}=x+c$

Eg. Solve $y'=(-2x+y)^2-7$, $y(0)=0$. [2010成大電研]

(Sol.) Let $u=-2x+y$, $\frac{du}{dx}+2=u^2-7$, $\frac{du}{dx}=u^2-9$, $\frac{du}{u^2-9}=\frac{1}{6}[\frac{1}{u-3}-\frac{1}{u+3}]du=dx$,

$ln(\frac{u-3}{u+3})=6x+c$, $\frac{u-3}{u+3}=Ae^{6x}$, $\frac{y-2x-3}{y-2x+3}=Ae^{6x}$, $y(0)=0 \Rightarrow A=-1$, $\therefore\ \frac{y-2x-3}{y-2x+3}=-e^{6x}$

Eg. Solve $y'=y^2-2xy+x^2+1$. [1991中央資電所、交大控制所]

(Sol.) Let $u=y-x$, $y'=u'+1=u^2+1$, $\frac{du}{u^2}=dx$, $-1/u=x+c \Rightarrow \frac{-1}{y-x}=x+c$

Eg. Solve $1+x^2y^2+y+xy'=0$. [交大電信所]

(Sol.) Let $u=xy$, $u'=y+xy'$, $1+u^2+u'=0$, $\frac{du}{1+u^2}=-dx$, $\tan^{-1}(u)=-x+c \Rightarrow \tan^{-1}(xy)=-x+c$

Eg. Solve $xy''+2y'=4x^3$.

(Sol.) Let $u=y' \Rightarrow xu'+2u=4x^3 \Rightarrow u=\frac{4x^3}{5}+\frac{c}{x^2} \Rightarrow y(x)=\int u(x)dx=\frac{x^4}{5}-\frac{c}{x}+D$

Another method: $x^2y''+2xy'=4x^4$ (Euler's equation)

Eg. Solve *y*”-2*yy*’=0.

(Sol.) Set *u*=*y*’, $y''=\frac{dy'}{dx}=\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}=u\frac{du}{dy}\Rightarrow u\frac{du}{dy}-2yu=0\Rightarrow u=y^2+c$

$$\Rightarrow\frac{dy}{dx}=y^2+c\Rightarrow\frac{dy}{y^2+c}=dx\Rightarrow\frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{y}{\sqrt{c}}\right)=x+K$$

Eg. Solve $\frac{dy}{dx}=\frac{x}{x^2y+y^3}$.

(Sol.) Let *u*=x^2, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=2x\cdot\frac{dy}{du}=\frac{x}{uy+y^3}$

$$\frac{du}{dy}-2yu=2y^3\Rightarrow[u\cdot e^{-y^2}]'=2y^3e^{-y^2}$$

$$u=e^{+y^2}\cdot[-y^2e^{-y^2}-e^{-y^2}+c]=-y^2-1+ce^{y^2}\Rightarrow x^2=-y^2-1+ce^{y^2}$$

Another method: $\frac{dx}{dy}=xy+\frac{y^3}{x}$ (Bernoulli’s equation)

Given a solution $y_1(x)$ of $y''+P(x)y'+Q(x)y=0$, then a second solution $y_2(x)=v(x)y_1(x)$ is obtained by the following method:

$v''y_1(x)+v'[2y_1'(x)+P(x)y_1]=0$. Set *v*’=*u*$\Rightarrow u'+\left[\frac{2y_1'}{y_1}+P(x)\right]u=0$

Eg. Solve $y''+\frac{3}{x}y'+\frac{1}{x^2}y=0$ **if** $y_1=\frac{1}{x}$ **is given.**

(Sol.) Let $y_2=v(x)y_1(x)=\frac{v(x)}{x}$, $y_2'=v'\cdot y_1+v\cdot y_1'$

$$y_2''=(v'\cdot y_1+v\cdot y_1')'=v''\cdot y_1+v'\cdot y_1'+v'\cdot y_1'+vy_1''=v''\cdot y_1+2v'\cdot y_1'+vy_1''$$

$$v''\cdot y_1+2v'\cdot y_1'+vy_1''+\frac{3}{x}(v'\cdot y_1+vy_1')+\frac{vy_1}{x^2}=v''y_1+2v'y_1'+\frac{3}{x}v'y_1=0$$

$$v'=u\Rightarrow u'+\left(\frac{2y_1'}{y_1}+\frac{3}{x}\right)u=u'+\left(\frac{-2\cdot\frac{1}{x^2}}{\frac{1}{x}}+\frac{3}{x}\right)u=u'+\frac{1}{x}u=0$$

$$u=v'=\frac{1}{x}\Rightarrow v(x)=\ell n\,|\,x\,|\Rightarrow y_2(x)=\frac{\ell n\,|\,x\,|}{x}$$

Eg. Given *y*(*x*)=*x* is a solution of *y*”-*xy*’+*y*=0, find the other solution. [交大電信所]