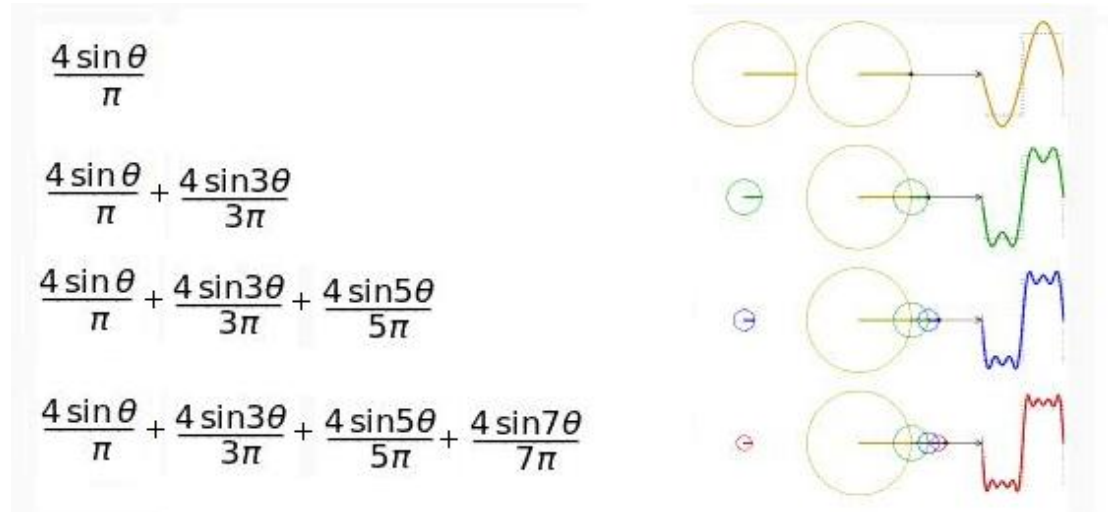


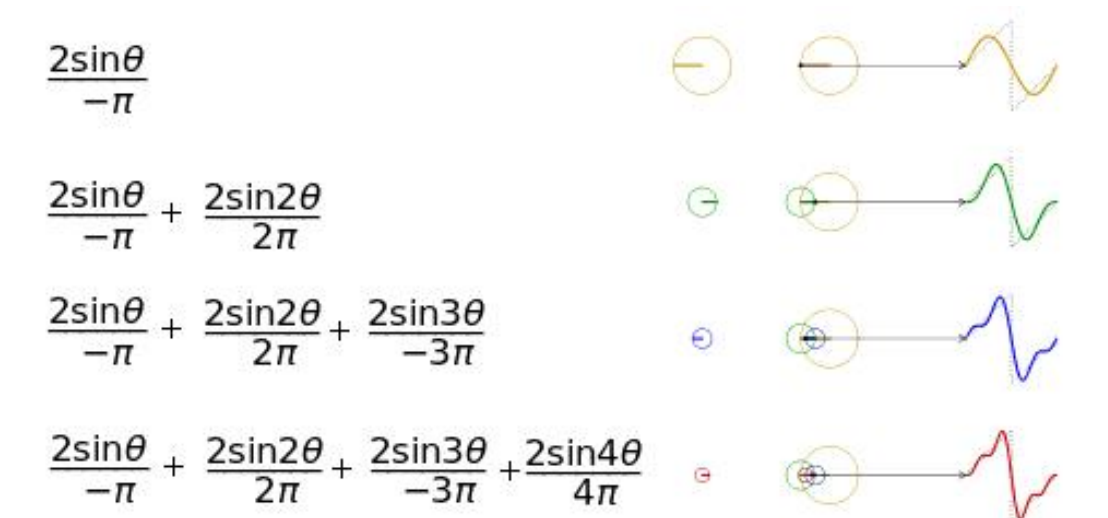
Chapter 5 Fourier Analysis

5-1 Fourier Series of a Periodical Function

Video Example of Fourier Series of Periodical Rectangular Function:



Video Example of Fourier Series of Periodical Sawtooth Function:



Fourier series: $f(x)$ is a periodical function with period= $2L$ and defined on an interval:

$-L \leq x \leq L$. $f(x+2L)=f(x)$, and then $f(x)=\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$, where

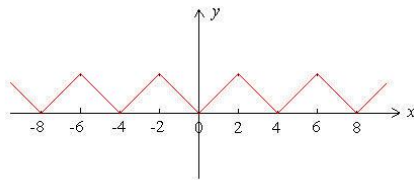
$$\frac{a_0}{2} = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad \text{and} \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

In case $f(x)$ is $\begin{matrix} \text{odd} \\ \text{even} \end{matrix} \Rightarrow \begin{matrix} a_n=0 \\ b_n=0 \end{matrix}$

Parseval's Identity for Fourier series: $\frac{1}{L} \int_{-L}^L |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$

Orthogonalities:

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, n \neq m \\ L, n = m \end{cases} \quad \text{and} \quad \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, n \neq m \\ L, n = m \end{cases}$$



Eg. Expand $f(x)=\begin{cases} x, & 0 \leq x \leq 2 \\ -x, & -2 \leq x \leq 0 \end{cases}$, $f(x+4)=f(x)$

into Fourier series and $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = ?$

(Sol.) $f(x) = \begin{cases} x, & 0 \leq x \leq 2 \\ -x, & -2 \leq x \leq 0 \end{cases}$, $f(x+4)=f(x)$, $2L = 4$, $L = 2$,

\therefore Even function, $\therefore b_n=0$, $\frac{a_0}{2} = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{4} \left[\int_{-2}^0 -x dx + \int_0^2 x dx \right] = 1$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \left[\int_{-2}^0 -x \cos\left(\frac{n\pi x}{2}\right) dx + \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx \right] \\ &= \frac{2}{2} \cdot \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx = \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2 \pi^2} \left(-\cos\left(\frac{n\pi x}{2}\right) \right) \right] \Bigg|_0^2 \\ &= \frac{4}{n^2 \pi^2} [\cos(n\pi) - 1] = \begin{cases} \frac{-8}{n^2 \pi^2}, & n : \text{odd} \\ 0, & n : \text{even} \end{cases} = \frac{-8}{(2m-1)^2 \pi^2}, m = 1, 2, \dots \end{aligned}$$

$$\therefore f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (\cos n\pi - 1) \cos\left(\frac{n\pi x}{2}\right) = 1 - \frac{8}{\pi^2} \left(\cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \dots \right)$$

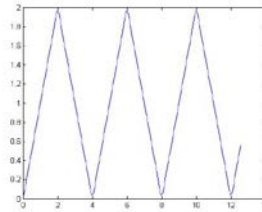
$$f(0) = 0 = 1 - \frac{8}{\pi^2} \left(\cos 0 + \frac{1}{3^2} \cos 0 + \frac{1}{5^2} \cos 0 + \dots \right)$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

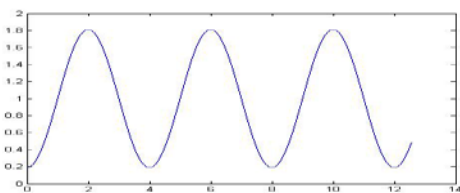
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$$1 + \sum_{n=1}^{10} \frac{4}{n^2 \pi^2} (\cos n\pi - 1) \cos\left(\frac{n\pi x}{2}\right) = 1 - \sum_{i=1}^5 \frac{8}{(2i-1)^2 \pi^2} \cos\left(\frac{(2i-1)\pi x}{2}\right).$$

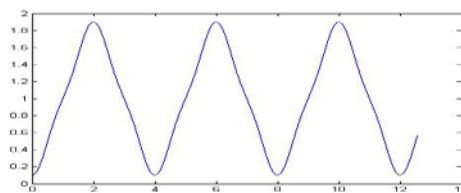
```
>>x = 0:0.001:4*pi; y=1;
>>for i=1:5
y=y-8*cos((2*i-1)*pi*x/2)/(2*i-1)^2/pi^2
end
>>plot (x,y)
```



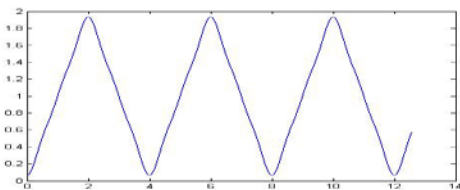
$$1 - \frac{8}{\pi^2} \cos\left(\frac{\pi x}{2}\right)$$



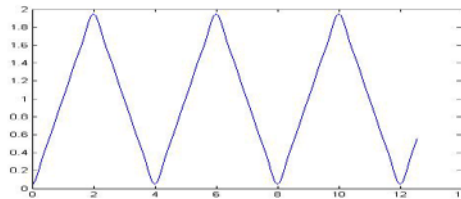
$$1 - \frac{8}{\pi^2} \left(\cos\frac{\pi x}{2} + \frac{1}{3^2} \cos\frac{3\pi x}{2} \right)$$



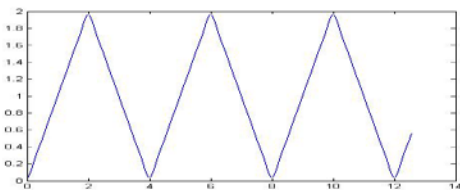
$$1 - \frac{8}{\pi^2} \left(\cos\frac{\pi x}{2} + \frac{1}{3^2} \cos\frac{3\pi x}{2} + \frac{1}{5^2} \cos\frac{5\pi x}{2} \right)$$



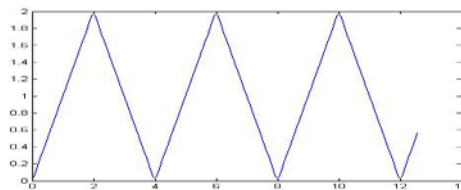
$$1 - \frac{8}{\pi^2} \left(\cos\frac{\pi x}{2} + \frac{1}{3^2} \cos\frac{3\pi x}{2} + \frac{1}{5^2} \cos\frac{5\pi x}{2} + \frac{1}{7^2} \cos\frac{7\pi x}{2} \right)$$

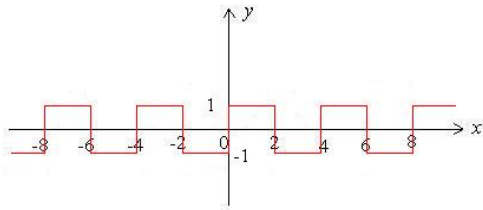


$$1 - \frac{8}{\pi^2} \left(\cos\frac{\pi x}{2} + \frac{1}{3^2} \cos\frac{3\pi x}{2} + \frac{1}{5^2} \cos\frac{5\pi x}{2} + \frac{1}{7^2} \cos\frac{7\pi x}{2} + \frac{1}{9^2} \cos\frac{9\pi x}{2} \right)$$



$$1 + \sum_{n=1}^{22} \frac{4}{n^2 \pi^2} (\cos n\pi - 1) \cos\left(\frac{n\pi x}{2}\right)$$





**Eg. Expand $f(x) = \begin{cases} 1, & 0 \leq x \leq 2 \\ -1, & -2 \leq x \leq 0 \end{cases}$ and $f(x+4) = f(x)$ into Fourier series. Find (a) $\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)}$ and (b) $\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}$. [文
化電機轉學考]**

(Sol.) $2L = 4, L = 2, f(x) = \begin{cases} 1, & 0 \leq x \leq 2 \\ -1, & -2 \leq x \leq 0 \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \because \text{Odd function, } \therefore a_n = 0, \forall n$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \left[\int_{-2}^0 -\sin\left(\frac{n\pi x}{2}\right) dx + \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx \right]$$

$$= \frac{2}{n\pi} - \frac{2 \cos(n\pi)}{n\pi} = \frac{2}{n\pi} [1 - \cos(n\pi)] \Rightarrow f(x) = \sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$

(a) Set $x=1$,

$$f(1) = 1 = \sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi}{2}\right) \right\} = \frac{4}{\pi} \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right\}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} = \frac{\pi}{4}$$

(b) $\frac{1}{L} \int_{-L}^L |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2],$

$$\frac{1}{2} \int_{-2}^2 1^2 dx = 2 = \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} [1 - \cos(n\pi)]^2 = \frac{16}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$$

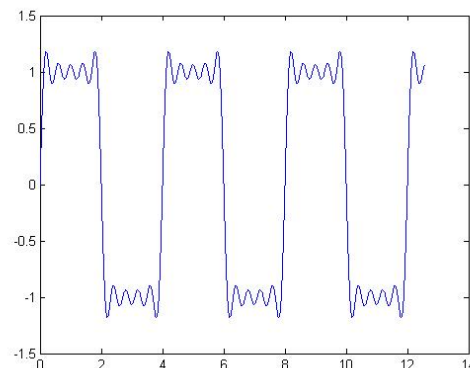
$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$$

In **Matlab** language, we can use the following instructions to obtain the finite sum of

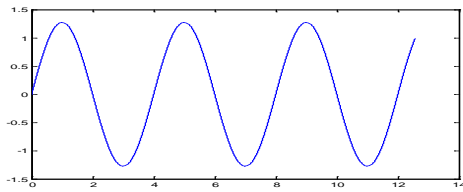
$$\sum_{n=1}^{10} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$

$$= \sum_{i=1}^5 \left\{ \frac{4}{(2i-1)\pi} \cdot \sin\left(\frac{(2i-1)\pi x}{2}\right) \right\}.$$

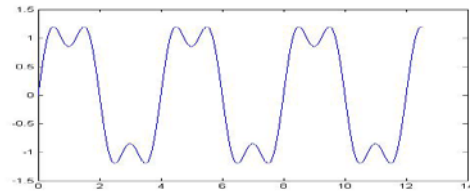
```
>>x = 0:0.001:4*pi;
>>y=0;
>>for i=1:5
y=y+4*sin((2*i-1)*pi*x/2)/(2*i-1)/pi
end
>>plot (x,y)
```



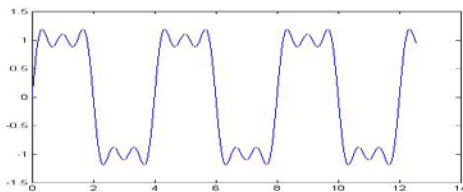
$$\frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right)$$



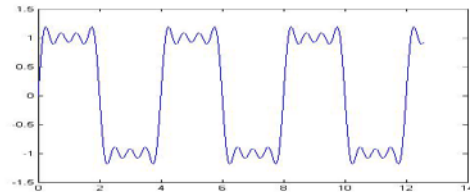
$$\sum_{n=1}^4 \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



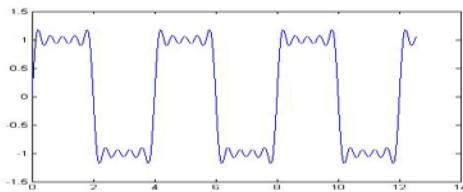
$$\sum_{n=1}^6 \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



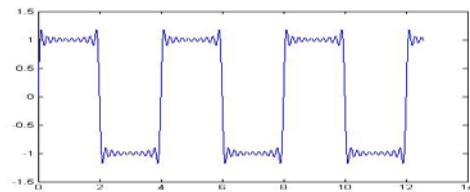
$$\sum_{n=1}^8 \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



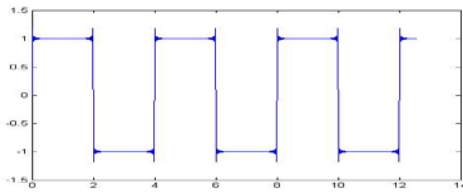
$$\sum_{n=1}^{10} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



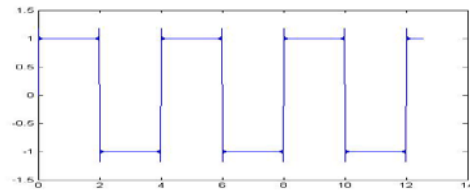
$$\sum_{n=1}^{20} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



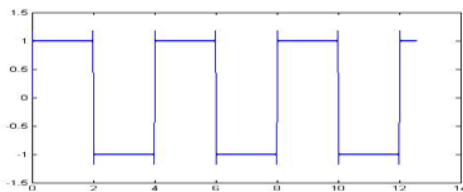
$$\sum_{n=1}^{200} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



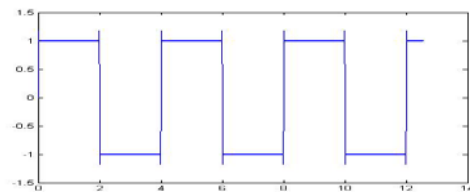
$$\sum_{n=1}^{400} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$

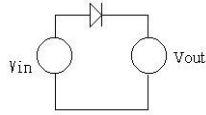
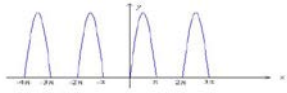


$$\sum_{n=1}^{1000} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$



$$\sum_{n=1}^{2000} \left\{ \frac{2}{n\pi} [1 - \cos(n\pi)] \cdot \sin\left(\frac{n\pi x}{2}\right) \right\}$$





Eg. Find the Fourier series of
 $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$ **and use the**

results to show that $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \times 3} - \frac{1}{3 \times 5} + \frac{1}{5 \times 7} - \frac{1}{7 \times 9} + \dots$. [2004 台大電研]

(Sol.)

$$(a) \quad 2L = 2\pi, \quad L = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$\frac{a_0}{2} = \frac{1}{2\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} \sin x dx \right] = \frac{1}{2\pi} [1 - \cos \pi] = \frac{1}{\pi}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cdot \cos(nx) dx + \int_0^{\pi} \sin(x) \cdot \cos(nx) dx \right] = \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(x+nx) + \sin(x-nx)] dx$$

$$= \frac{1}{2\pi} \left[\frac{1 - \cos(1+n)\pi}{1+n} + \frac{1 - \cos(1-n)\pi}{1-n} \right]$$

$$= \frac{1}{2\pi} \left[\left(\frac{1}{1+n} + \frac{1}{1-n} \right) - \frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} \right] = \frac{1}{2\pi} \left[\frac{2}{1-n^2} + \frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} \right]$$

$$= \frac{1}{2\pi} \left[\frac{2}{1-n^2} + \frac{2 \cos n\pi}{1-n^2} \right] = \frac{1}{2\pi} \cdot \frac{2}{1-n^2} (1 + \cos n\pi)$$

$$= \frac{1 + \cos n\pi}{\pi(1-n^2)} (n \neq 1) = \begin{cases} 0, & \forall n = 3, 5, 7, \dots \\ \frac{2}{\pi(1-n^2)}, & \forall n = 2, 4, 6, \dots \end{cases}$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cdot \sin(nx) dx + \int_0^{\pi} \sin(x) \cdot \sin(nx) dx \right]$$

$$= \frac{1}{\pi} \left\{ \int_0^{\pi} \frac{1}{2} [\cos(x-nx) - \cos(x+nx)] dx \right\} = \frac{1}{2\pi} \left[\frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi}{1+n} \right] = \begin{cases} 1/2, & n = 1 \\ 0, & n > 1 \end{cases}$$

$$\therefore f(x) = \frac{1}{\pi} + \frac{2}{\pi} \left[\frac{\cos(2x)}{-3} + \frac{\cos(4x)}{-15} + \frac{\cos(6x)}{-35} + \frac{\cos(8x)}{-63} + \dots \right] + \frac{1}{2} \sin(x)$$

$$(b) \quad f\left(-\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + \frac{2}{\pi} \left[\frac{\cos(-\pi)}{-3} + \frac{\cos(-2\pi)}{-15} + \frac{\cos(-3\pi)}{-35} + \frac{\cos(-4\pi)}{-63} + \dots \right] - \frac{1}{2}$$

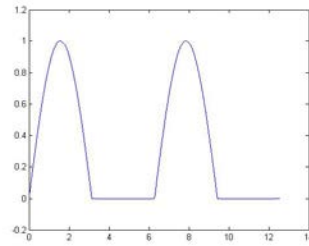
$$\frac{1}{2} = \frac{1}{\pi} + \left[\frac{2}{3\pi} - \frac{2}{15\pi} + \frac{2}{35\pi} - \frac{2}{63\pi} + \dots \right] = \frac{2}{\pi} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} \dots \right)$$

$$\therefore \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \times 3} - \frac{1}{3 \times 5} + \frac{1}{5 \times 7} - \frac{1}{7 \times 9} + \dots$$

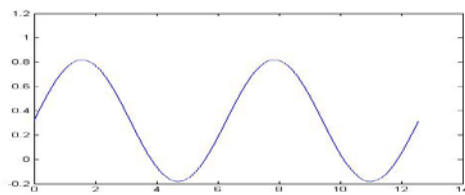
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$$\frac{1}{\pi} - \frac{2}{\pi} \left[\frac{\cos(2x)}{3} + \frac{\cos(4x)}{15} + \frac{\cos(6x)}{35} + \frac{\cos(8x)}{63} + \dots + \frac{\cos(40x)}{1599} \right] + \frac{1}{2} \sin(x).$$

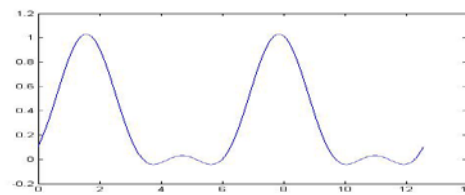
```
>>x = 0:0.001:4*pi; y=1/pi+sin(x)/2;
>>for n=1:20
y=y-2*cos(2*n*x)/pi/(4*n^2-1)
end
>>plot (x,y)
```



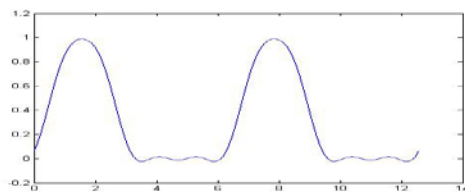
$$\frac{1}{\pi} + \frac{1}{2} \sin(x)$$



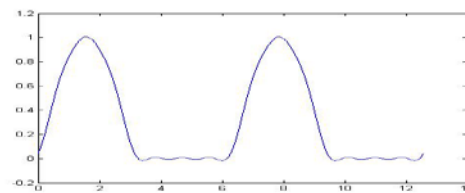
$$\frac{1}{\pi} - \frac{2}{\pi} \cdot \frac{\cos(2x)}{3} + \frac{1}{2} \sin(x)$$



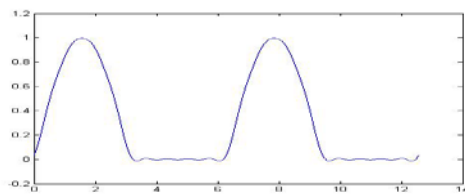
$$\frac{1}{\pi} - \frac{2}{\pi} \left[\frac{\cos(2x)}{3} + \frac{\cos(4x)}{15} \right] + \frac{1}{2} \sin(x)$$



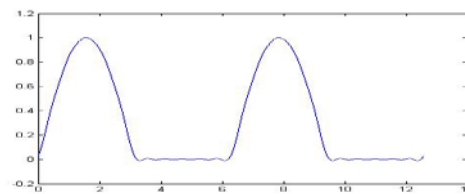
$$\frac{1}{\pi} - \frac{2}{\pi} \left[\frac{\cos(2x)}{3} + \frac{\cos(4x)}{15} + \frac{\cos(6x)}{35} \right] + \frac{1}{2} \sin(x)$$



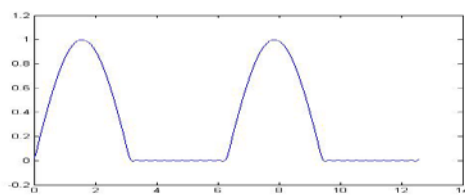
$$\frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^4 \frac{\cos(2nx)}{4n^2 - 1} + \frac{1}{2} \sin(x)$$



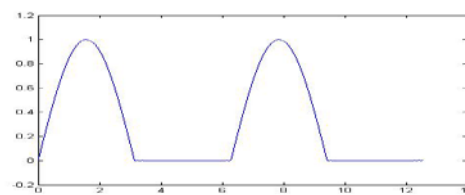
$$\frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^5 \frac{\cos(2nx)}{4n^2 - 1} + \frac{1}{2} \sin(x)$$



$$\frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{10} \frac{\cos(2nx)}{4n^2 - 1} + \frac{1}{2} \sin(x)$$



$$\frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{20} \frac{\cos(2nx)}{4n^2 - 1} + \frac{1}{2} \sin(x)$$



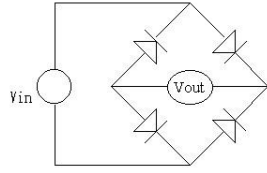
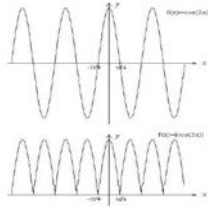


Fig. Find the Fourier series of $|\cos(2x)|$ and calculate $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$ [1990 交大材研、成大電研]

(Sol.) (a) $f(x) = f\left(x + \frac{\pi}{2}\right)$, $f(x) = \cos(2x)$, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

$$2L = \frac{\pi}{2}, L = \frac{\pi}{4}, \frac{n\pi x}{L} = 4nx, \frac{a_0}{2} = \frac{1}{2L} \int_{-L}^L \cos(2x) dx = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{L} \int_{-L}^L \cos(2x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx = \frac{4}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(2x) \cos(4nx) dx$$

$$= \frac{4}{\pi} \cdot \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} [\cos((2+4n)x) + \cos((2-4n)x)] dx$$

$$= \frac{2}{\pi} \cdot \left[\frac{\sin(2+4n)x}{2+4n} \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \frac{\sin(2-4n)x}{2-4n} \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right]$$

$$= \frac{2}{\pi} \cdot \left[\frac{2 \sin \frac{(2n+1)\pi}{2}}{2(2n+1)} + \frac{2 \sin \frac{(2n-1)\pi}{2}}{2(2n-1)} \right] = \frac{2}{\pi} \cdot \left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1} \right]$$

$$= \frac{2}{\pi} \cdot \left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1} \right] = -\frac{4}{\pi} \cdot \frac{(-1)^n}{4n^2 - 1} = \frac{4}{\pi} \cdot \frac{(-1)^{n+1}}{4n^2 - 1}$$

\therefore Even function, $\therefore b_n = 0 \Rightarrow f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx) = |\cos(2x)|$

(b) $x = 0, f(x) = 1 = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} = \frac{\pi}{4} \left(1 - \frac{2}{\pi}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{\pi}{4} \left(\frac{2}{\pi} - 1\right)$$

In **Matlab** language, we can use the following instructions to obtain the finite sum of

$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^6 \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx).$$

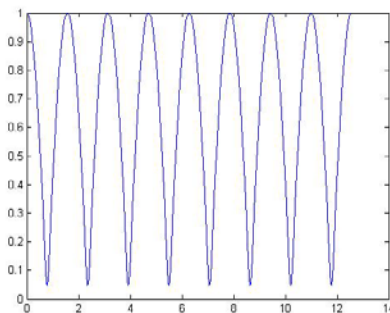
`>>x = 0:0.001:4*pi; y=2/pi;`

`>>for n=1:6`

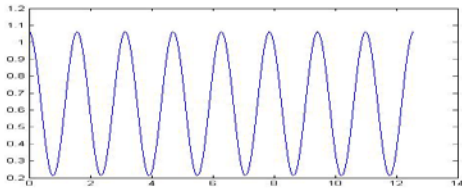
`y=y+4*(-1)^(n+1)*cos(4*n*x)/(4*n^2-1)/pi`

`end`

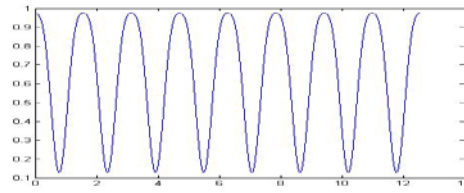
`>>plot(x,y)`



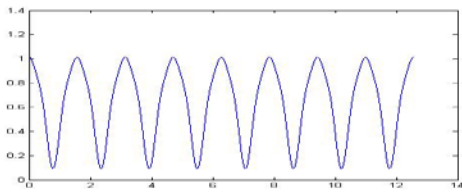
$$\frac{2}{\pi} + \frac{4}{3\pi} \cos(4x)$$



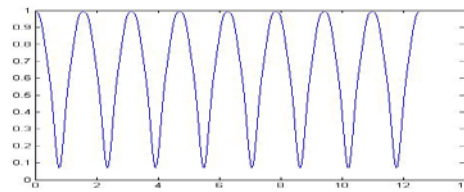
$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^2 \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$



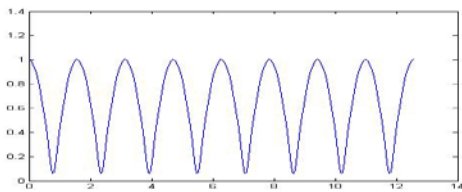
$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^3 \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$



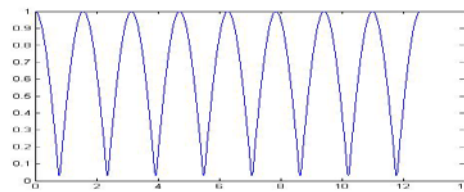
$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^4 \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$



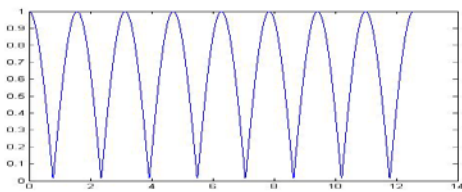
$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^5 \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$



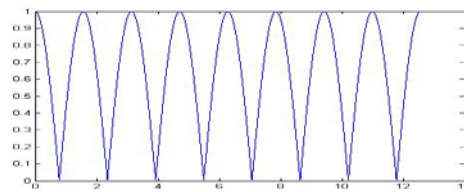
$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^6 \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$



$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{10} \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$

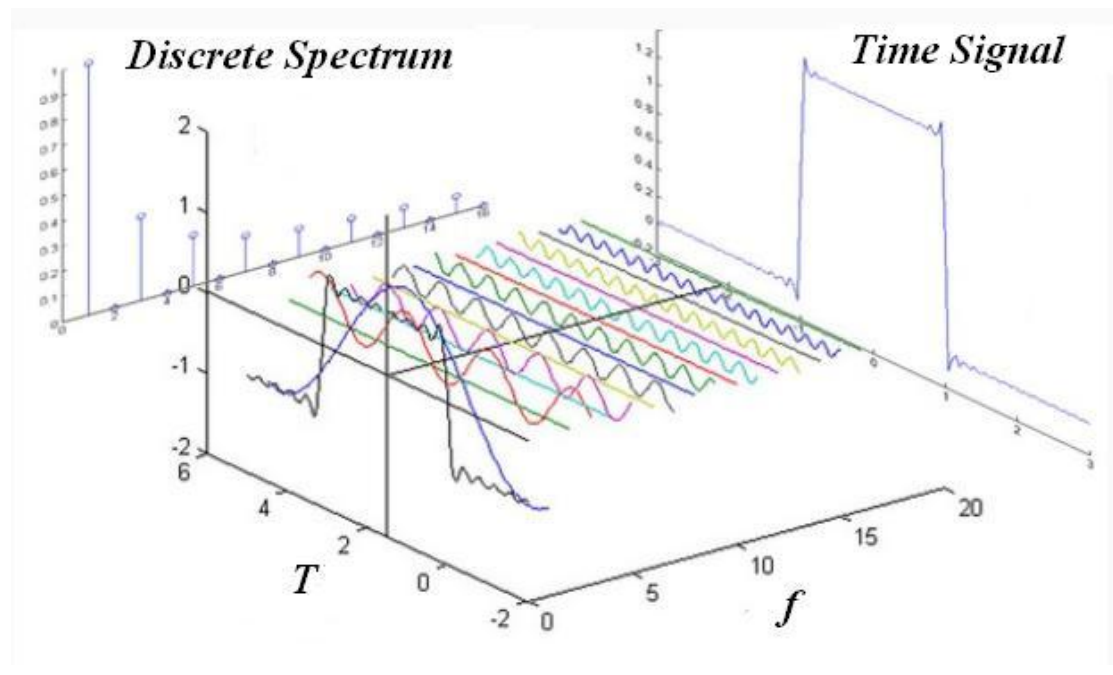
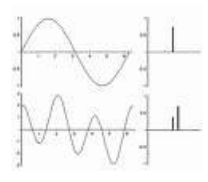


$$\frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{100} \frac{(-1)^{n+1}}{4n^2 - 1} \cdot \cos(4nx)$$



Discrete spectrum of $f(t)$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right] = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{n\pi t}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}$$



5-2 Fourier Transforms and Inverse Fourier Transforms

Fourier Transform pair defined in Engineering:

$$\begin{cases} F(\omega) = \mathfrak{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x)e^{-j\omega x} dx \\ f(x) = \mathfrak{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega x} d\omega \end{cases}, \text{ where } \omega = 2\pi v.$$

Parseval's Identities for Fourier Transform pairs $\mathfrak{F}[f(x)]=F(\omega)$ and $\mathfrak{F}[g(x)]=G(\omega)$:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)g^*(x)dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$

Continuous Spectrum of $f(t)$: $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$

Convolution in Fourier Transform: $f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

Correlation in Fourier Transform: $f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)g(\tau-t)d\tau$

Basic theorems of Fourier Transforms $\mathfrak{F}[f(x)]=F(\omega)$ and $\mathfrak{F}[g(x)]=G(\omega)$:

1. $\mathfrak{F}[af(x)+bg(x)]=aF(\omega)+bG(\omega)$

2. $\mathfrak{F}[f(ax)]=[F(\omega/a)]/a$ and $\mathfrak{F}^{-1}[F(a\omega)]=[f(x/a)]/a, a>0$

(Proof) For $a>0$, let $ax=u$

$$\begin{aligned} \mathfrak{F}[f(ax)] &= \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(ax)dx = \frac{1}{a} \int_{-\infty}^{\infty} e^{-j(\frac{\omega}{a})ax} \cdot f(ax)d(ax) = \frac{1}{a} \int_{-\infty}^{\infty} e^{-j(\frac{\omega}{a})u} \cdot f(u)du \\ &= \frac{1}{a} F\left(\frac{\omega}{a}\right) \end{aligned}$$

3. $\mathfrak{F}[f(x)e^{j\omega_0 x}]=F(\omega-\omega_0)$ and $\mathfrak{F}^{-1}[F(\omega-\omega_0)]=f(x)e^{j\omega_0 x}$

(Proof) $\mathfrak{F}[f(x)e^{j\omega_0 x}] = \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x)e^{j\omega_0 x} dx = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)x} \cdot f(x)dx = F(\omega-\omega_0)$

4. $\mathfrak{F}[f'(x)]=j\omega F(\omega)$, $\mathfrak{F}[f^{(n)}(x)]=j\omega^n F(\omega)$ in case of $f(\pm\infty)=f'(\pm\infty)=f''(\pm\infty)=\dots=0$

(Proof) $\mathfrak{F}[f'(x)] = \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f'(x)dx = \int_{-\infty}^{\infty} e^{-j\omega x} df(x)$

$$= e^{-j\omega x} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-j\omega) e^{-j\omega x} \cdot f(x)dx = e^{-j\omega\infty} f(\infty) - e^{-j\omega(-\infty)} f(-\infty) + j\omega \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x)dx = j\omega F(\omega)$$

By mathematical induction, we have $\mathfrak{F}[f^{(n)}(x)]=j\omega^n F(\omega)$ if

$$f(\pm\infty)=f'(\pm\infty)=f''(\pm\infty)=\dots=0.$$

5. $\mathfrak{F}[f(x)*g(x)]=F(\omega)G(\omega)$

6. $\mathfrak{F}[g^*(x) \star f(x)]=F(\omega)G^*(\omega)$, where $g^*(x)$ and $G^*(\omega)$ are the complex conjugates of $g(x)$ and $G(\omega)$, respectively.

7. $\mathfrak{F}[f(x-a)] = e^{-ja\omega}F(\omega)$ and $\mathfrak{F}^{-1}[e^{-ja\omega}F(\omega)] = f(x-a)$

(Proof) Let $x-a=u$, $\mathfrak{F}[f(x-a)] = \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x-a)dx = \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x-a)d(x-a)$
 $= e^{-ja\omega} \int_{-\infty}^{\infty} e^{-j\omega(x-a)} \cdot f(x-a)d(x-a) = e^{-ja\omega} \int_{-\infty}^{\infty} e^{-j\omega u} \cdot f(u)du$
 $= e^{-ja\omega}F(\omega)$

8. $\mathfrak{F}[x^n f(x)] = (j)^n F^{(n)}(\omega)$

(Proof) $\mathfrak{F}[xf(x)] = \int_{-\infty}^{\infty} e^{-j\omega x} \cdot xf(x)dx = \int_{-\infty}^{\infty} j \frac{d}{d\omega} e^{-j\omega x} \cdot f(x)dx = j \frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x)dx$
 $= jF'(\omega)$

By mathematical induction, we have $\mathfrak{F}[x^n f(x)] = (j)^n F^{(n)}(\omega)$.

9. $\mathfrak{F}[\delta(x)] = 1$ and $\mathfrak{F}[1] = 2\pi\delta(\omega)$

Eg. Find (a) $\mathfrak{F}[xe^{-|x|}]$, (b) $\mathfrak{F}[e^{-3|x|}]$, (c) $\mathfrak{F}^{-1}\left\{\frac{4}{4+\omega^2}\right\}$, (d) $\int_{-\infty}^{\infty} \frac{\cos \omega}{\omega^2 + 4} d\omega$, (e) $f(x)$

if $\int_0^{\infty} f(x)\cos(2x)dx = e^{-2}$. [文化電機轉學考]

(Sol.) $\mathfrak{F}[e^{-a|x|}] = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx = \int_{-\infty}^0 e^{ax} e^{-i\omega x} dx + \int_0^{\infty} e^{-ax} e^{-i\omega x} dx$
 $= \frac{e^{(a-i\omega)x}}{a-i\omega} \Big|_{-\infty}^0 + \frac{e^{-(a-i\omega)x}}{-a-i\omega} \Big|_0^{\infty} = \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2 + \omega^2}$

(a) According to $\mathfrak{F}[x^n f(x)] = (j)^n \frac{d^n}{d\omega^n} F(\omega)$,

$$\mathfrak{F}[x \cdot e^{-a|x|}] = j \frac{d}{d\omega} \mathfrak{F}[e^{-a|x|}] = j \frac{d}{d\omega} \left(\frac{2a}{a^2 + \omega^2} \right) = \frac{-4ai\omega}{(a^2 + \omega^2)^2}$$

$$a = 1, \mathfrak{F}[xe^{-|x|}] = \frac{-i4\omega}{(\omega^2 + 1)^2}$$

(b) $a = 3, \mathfrak{F}[e^{-3|x|}] = \frac{6}{\omega^2 + 9}$, (c) $a = 2, \mathfrak{F}^{-1}\left[\frac{4}{4 + \omega^2}\right] = e^{-2|x|}$

(d) $\mathfrak{F}^{-1}\left[\frac{4}{4 + \omega^2}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{4 + \omega^2} e^{i\omega x} d\omega = e^{-2|x|}$

$$x = 1, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{4 + \omega^2} e^{i\omega} d\omega = e^{-2}, \quad \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{4 + \omega^2} [\cos \omega + i \sin \omega] d\omega = e^{-2}$$

$$I_m(\dots) = 0, R_e(\dots) \neq 0, \therefore \int_{-\infty}^{\infty} \frac{\cos \omega}{4 + \omega^2} d\omega = \frac{\pi}{2} e^{-2}$$

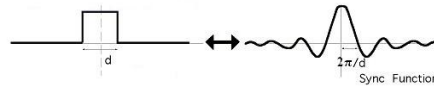
(e) $\int_{-\infty}^{\infty} \frac{2}{\pi} \cdot \frac{\cos \omega}{\omega^2 + 4} d\omega = e^{-2} = 2 \int_0^{\infty} \frac{2}{\pi} \cdot \frac{\cos \omega}{\omega^2 + 4} d\omega = \int_0^{\infty} \frac{4}{\pi} \frac{\cos \omega}{\omega^2 + 4} d\omega$

Set $\omega = 2x, \int_0^{\infty} \frac{4}{\pi} \cdot \frac{\cos 2x}{4x^2 + 4} \cdot 2dx = e^{-2} = \int_0^{\infty} f(x)\cos(2x)dx, \therefore f(x) = \frac{2}{\pi} \cdot \frac{1}{(x^2 + 1)}$

Eg. For two rectangular functions: $f(x)=\begin{cases} 1, & |x|\leq 1 \\ 0, & |x|>1 \end{cases}$, $g(x)=\begin{cases} 1, & |x|\leq 2 \\ 0, & |x|>2 \end{cases}$, find (a)

$\mathfrak{T}[f(x)]$, (b) $\mathfrak{T}[g(x)]$, (c) $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$ [文化電機轉學考], and $\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega$ [2008 成大電研].

(Sol.) (a) $\mathfrak{T}[f(x)] = \int_{-1}^1 e^{-i\omega x} dx = \frac{e^{-i\omega x}}{-i\omega} \Big|_{-1}^1 = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} = \frac{2(e^{i\omega} - e^{-i\omega})}{2i\omega} = \frac{2\sin(\omega)}{\omega}$

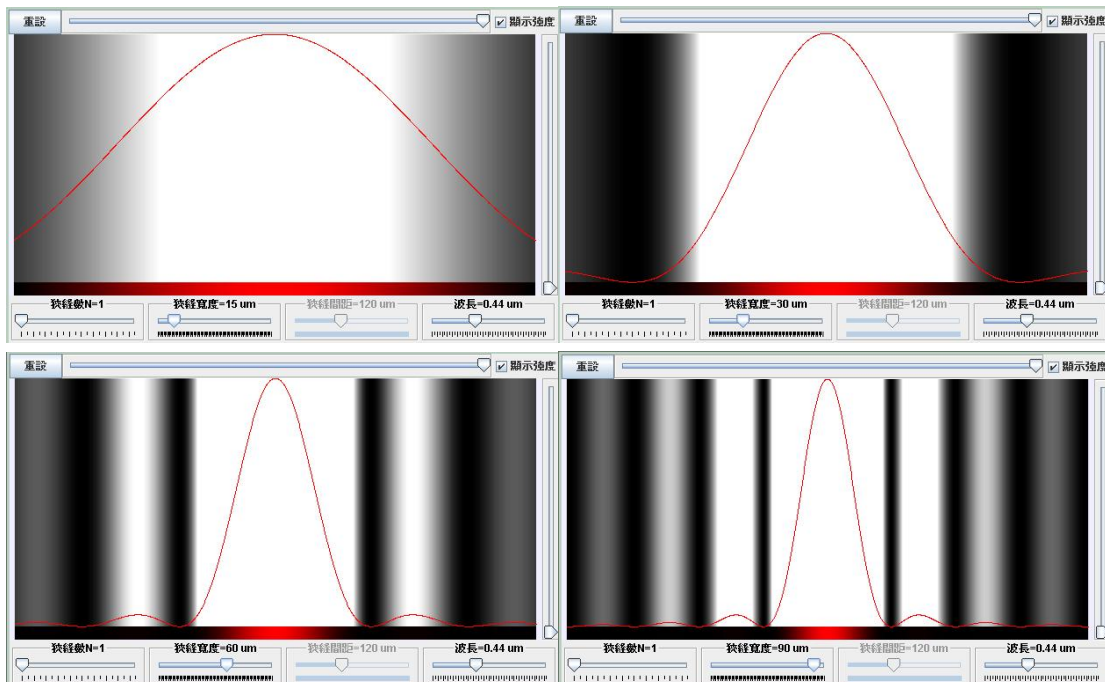
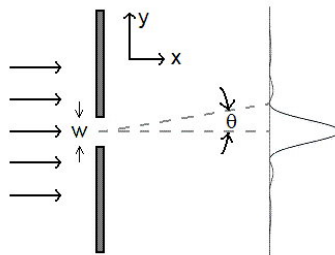


(b) $\because \mathfrak{T}[f(ax)] = \frac{1}{a} F\left(\frac{\omega}{a}\right)$, $a>0$, $\mathfrak{T}[g(x)] = \mathfrak{T}\left[f\left(\frac{x}{2}\right)\right] = 2 \cdot \frac{2\sin(2\omega)}{2\omega} = \frac{2\sin(2\omega)}{\omega}$

(c) $f(x) = \mathfrak{T}^{-1}\left\{\frac{2\sin \omega}{\omega}\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin \omega}{\omega} e^{i\omega x} d\omega$, $f(0) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$

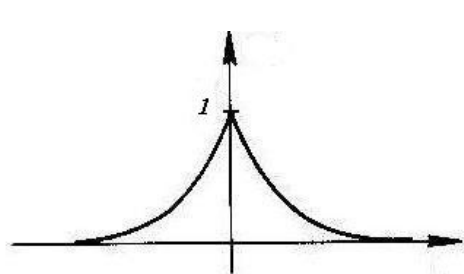
(d) According to $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$,

$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2\sin(\omega)}{\omega}\right)^2 d\omega = \int_{-1}^1 1^2 dx = 2 \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi$

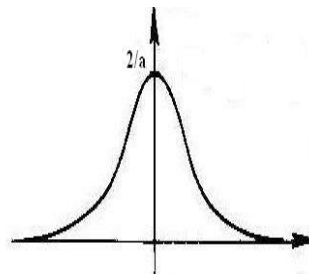


Eg. Find $\mathfrak{T}[e^{-a|x|}]$ and $\mathfrak{T}[e^{-|x|}]$.

$$\begin{aligned} \text{(Sol.) } \int_{-\infty}^{\infty} e^{-a|x|} \cdot e^{-j\omega x} dx &= \int_{-\infty}^0 e^{+ax} \cdot e^{-j\omega x} dx + \int_0^{\infty} e^{-ax} \cdot e^{-j\omega x} dx \\ &= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2 + \omega^2} = \mathfrak{T}[e^{-a|x|}]. \text{ For } a=1, \mathfrak{T}[e^{-|x|}] = \frac{2}{1+\omega^2} \end{aligned}$$



$$f(x) = e^{-a|x|}$$



$$F(\omega) = \frac{2a}{a^2 + \omega^2}$$

$$\text{Or, according to } \mathfrak{T}[f(ax)] = \frac{1}{a} F\left(\frac{\omega}{a}\right), \quad \mathfrak{T}[e^{-|x|}] = \mathfrak{T}\left[e^{-a\left(\frac{x}{a}\right)}\right] = a \cdot \frac{2a}{a^2 + (a\omega)^2} = \frac{2}{1+\omega^2}$$

Eg. Determine $\mathfrak{T}^{-1}\left\{\frac{1}{\omega^2 + 1}\right\}$. [2013 成大電研]

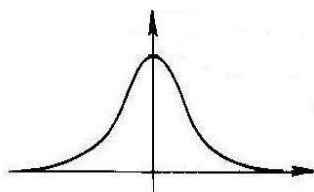
$$\text{(Sol.) } \mathfrak{T}^{-1}\left[\frac{2a}{\omega^2 + a^2}\right] = e^{-a|x|} \text{ and } a=1, \quad \mathfrak{T}^{-1}\left\{\frac{1}{\omega^2 + 1}\right\} = \frac{1}{2} \mathfrak{T}^{-1}\left\{\frac{2}{\omega^2 + 1}\right\} = \frac{1}{2} e^{-|x|}$$

Eg. Find $\mathfrak{T}\left[\frac{1}{a^2 + x^2}\right]$ and $\mathfrak{T}\left[\frac{1}{a^2 + (x+b)^2}\right]$.

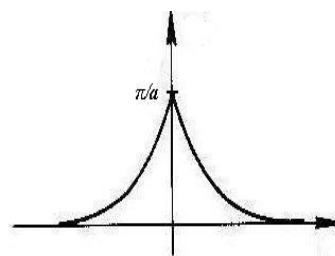
$$\begin{aligned} \text{(Sol.) } \mathfrak{T}[e^{-a|x|}] &= \frac{2a}{a^2 + \omega^2}, \quad e^{-a|x|} = \mathfrak{T}^{-1}\left[\frac{2a}{a^2 + \omega^2}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} \cdot e^{i\omega x} d\omega \\ &= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} e^{-i(-x)\omega} d\omega \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} e^{-i(-x)\omega} d\omega = \frac{\pi}{a} e^{-a|x|}. \text{ Set } u=-x \Rightarrow \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} \cdot e^{-i\omega u} d\omega = \frac{\pi}{a} e^{-a|u|}$$

$$\text{Set } x=\omega, \omega=u, \quad \mathfrak{T}\left[\frac{1}{a^2 + x^2}\right] = \int_{-\infty}^{\infty} \frac{1}{a^2 + x^2} e^{-i\omega x} dx = \frac{\pi}{a} \cdot e^{-a|\omega|}$$



$$f(x) = \frac{1}{a^2 + x^2}$$

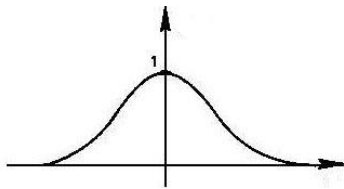


$$F(\omega) = \frac{\pi}{a} e^{-a|\omega|}$$

$$\therefore \mathfrak{T}[f(x-a)] = e^{-j\omega a} F(\omega), \quad \therefore \mathfrak{T}\left[\frac{1}{a^2 + (x+b)^2}\right] = e^{i\omega b} \cdot \frac{\pi}{a} e^{-a|\omega|}$$

Eg. Find $\mathfrak{F}(e^{-a^2x^2})$.

$$\begin{aligned}
 \text{(Sol.) } \int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{-j\omega x} dx &= \int_{-\infty}^{\infty} e^{-a^2\left(x^2 + j\frac{\omega}{a^2}x\right)} dx = e^{\frac{-\omega^2}{4a^2}} \cdot \int_{-\infty}^{\infty} e^{-a^2\left[x^2 + j\frac{\omega}{a^2}x - \frac{\omega^2}{4a^4}\right]} \cdot dx \\
 &= e^{\frac{-\omega^2}{4a^2}} \cdot \int_{-\infty}^{\infty} e^{-a^2\left[x + j\frac{\omega}{2a^2}\right]^2} \cdot dx = e^{\frac{-\omega^2}{4a^2}} \cdot \int_{-\infty}^{\infty} e^{-a^2u^2} \cdot du \leftarrow \left(u = x + j\frac{\omega}{2a^2}\right) \\
 &= \frac{\sqrt{\pi}}{a} e^{\frac{-\omega^2}{4a^2}} \left(\text{Note: } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a^2(u^2+v^2)} dudv = 4 \int_0^{\infty} \int_0^{\pi} e^{-a^2r^2} r dr d\theta = \frac{\pi}{a^2} \right)
 \end{aligned}$$

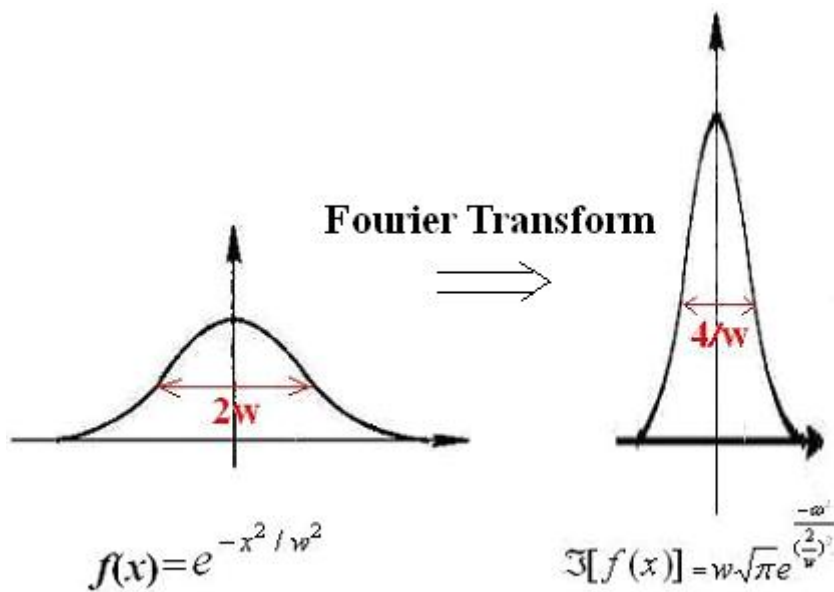


$$f(x) = e^{-a^2x^2}$$



$$F(\omega) = \frac{\sqrt{\pi}}{a} e^{\frac{-\omega^2}{4a^2}}$$

Note: $f(x) = \frac{1}{a^2 + x^2}$ and $g(x) = e^{-a^2x^2}$ are similar to each other. But their respective Fourier transforms look quite different!



$2w \cdot (4/w) = 8$ is a constant!

Eg. Determine $\mathfrak{F}\left[\frac{1}{a+jt}\right]$. [2003 台科大電研]

(Sol.) $\because \mathfrak{F}\left[\frac{1}{a^2+t^2}\right] = \int_{-\infty}^{\infty} \frac{1}{a^2+t^2} e^{-j\omega t} dt = \frac{\pi}{a} \cdot e^{-a|\omega|}$ and $\mathfrak{F}[t^n f(t)] = (j)^n \frac{d^n}{d\omega^n} F(\omega)$

$\therefore \mathfrak{F}\left[\frac{1}{a+jt}\right] = \mathfrak{F}\left[\frac{a}{a^2+t^2}\right] - \mathfrak{F}\left[\frac{jt}{a^2+t^2}\right] = a \cdot \frac{\pi}{a} e^{-a|\omega|} - j \cdot j \frac{d}{d\omega} \left[\frac{\pi}{a} e^{-a|\omega|}\right] = \pi e^{-a|\omega|} \cdot [1 - \text{sgn}(\omega)]$

$= \begin{cases} 0, & \omega \geq 0 \\ 2\pi e^{a\omega}, & \omega < 0 \end{cases} = 2\pi e^{a\omega} \cdot u(-\omega)$

Eg. For two functions: $f(t)=e^{-|t|}$ and $g(t)=\begin{cases} 1, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$, find $\mathfrak{F}[f(t)*g(t)]$. [2008 成大電研]

(Sol.) $\mathfrak{F}[f(t)*g(t)] = F(\omega)G(\omega) = \frac{2}{1+\omega^2} \cdot \frac{2\sin(\omega)}{\omega} = \frac{4\sin(\omega)}{\omega(\omega^2+1)}$

Fourier Transform pair defined in Mathematics:

$$\begin{cases} F(\omega) = \mathfrak{F}[f(x)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx \\ f(x) = \mathfrak{F}^{-1}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} d\omega \end{cases}, \text{ where } \omega = 2\pi\nu.$$

Fourier Transform pair defined in Physics/Optics:

$$\begin{cases} G(f) = \mathfrak{F}[g(x)] \equiv \int_{-\infty}^{\infty} g(x) e^{-j2\pi f x} dx \\ g(x) = \mathfrak{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi f x} df \end{cases}$$