

Chapter 6 Partial Differential Equations (PDE)

6-1 Classification of Partial Differential Equations

The first-order linear PDE: $a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} + f(x, y)u + g(x, y) = 0$

The second-order linear PDE: $a(x, y) \frac{\partial^2 u}{\partial x^2} + b(x, y) \frac{\partial^2 u}{\partial x \partial y} + c(x, y) \frac{\partial^2 u}{\partial y^2} + d(x, y) \frac{\partial u}{\partial x} + e(x, y) \frac{\partial u}{\partial y} + f(x, y)u + g(x, y) = 0$

$$\begin{cases} \text{hyperbolic at } (x_0, y_0): \Delta(x_0, y_0) = b(x_0, y_0)^2 - 4a(x_0, y_0)c(x_0, y_0) > 0 \\ \text{elliptic at } (x_0, y_0): \Delta(x_0, y_0) < 0 \\ \text{parabolic at } (x_0, y_0): \Delta(x_0, y_0) = 0 \end{cases}$$

Notations: $\frac{\partial^2 u}{\partial x^2} = u_{xx}$, $\frac{\partial^2 u}{\partial x \partial y} = u_{xy}$, $\frac{\partial u}{\partial x} = u_x$, etc.

Wave equation: $u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) + bu_t + F$

Heat equation or Diffusion equation: $u_t = a^2(u_{xx} + u_{yy} + u_{zz}) = a^2 \nabla^2 u$

Laplace's and Poisson's equations: $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \begin{cases} 0 \\ \rho \end{cases}$

Schrodinger's equation: $i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi$ in quantum mechanics.

6-2 Separation-of-Variable Method

Eg. Solve $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$, $\theta(x, 0) = x$, $\theta(0, t) = 0$, $\frac{\partial \theta}{\partial t}|_{t=0} = 0$, and $\frac{\partial \theta}{\partial x}|_{x=1} = 0$.

(Sol.) Let $\theta(x, t) = X(x)T(t)$, $X''(x)T(t) = X(x)T''(t)$, $\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$

$$X(0)=0, X'(1)=0 \Rightarrow -\lambda = \left[\frac{(2n-1)\pi}{2} \right]^2, \quad X_n = C_n \sin \left[\frac{(2n-1)\pi}{2} x \right]$$

$$\frac{T''(t)}{T(t)} = - \left[\frac{(2n-1)\pi}{2} \right]^2, \quad T(0)=\text{constant}, T'(0)=0 \Rightarrow T_n = d_n \cos \left[\frac{(2n-1)\pi}{2} t \right],$$

$$\therefore \theta(x, t) = \sum_{n=1}^{\infty} A_n \cos \left[\frac{(2n-1)\pi}{2} t \right] \cdot \sin \left[\frac{(2n-1)\pi}{2} x \right]$$

$$\theta(x, 0) = x = \sum_{n=1}^{\infty} A_n \sin \left[\frac{(2n-1)\pi}{2} x \right] \Rightarrow A_n = \frac{\int_{-1}^1 x \sin \left[\frac{(2n-1)\pi}{2} x \right] dx}{\int_{-1}^1 \sin^2 \left[\frac{(2n-1)\pi}{2} x \right] dx} = \frac{8(-1)^{n+1}}{(2n-1)^2 \pi^2}$$

Eg. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t)=u(1,t)=u(x,0)=0$, and $\frac{\partial u}{\partial t}|_{t=0} = \sin(\pi x)$.

(Sol.) $u(x,t) = X(x)T(t)$, $X''(x)T(t) = X(x)T''(t)$, $\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$,

$X(0)=0=X(1) \Rightarrow \lambda = -(n\pi)^2$ and $X(x) = C_n \sin(n\pi x)$,

$\frac{T''(t)}{T(t)} = -(n\pi)^2$ and $T(0)=0$, $T'(0)=\text{constant} \Rightarrow T(t) = d_n \sin(n\pi t)$,

$\therefore u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cdot \sin(n\pi t)$, $\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} n\pi A_n \sin(n\pi x) \cdot \cos(n\pi t)$

$\frac{\partial u}{\partial t}|_{t=0} = \sin(\pi x) \Rightarrow \pi A_1 = 1 \Rightarrow A_1 = 1/\pi$ but $A_n = 0$ for $n \neq 1 \Rightarrow u(x,t) = \frac{1}{\pi} \sin(\pi x) \cdot \sin(\pi t)$

Eg. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = 3\sin(2\pi x)$, $u(0,t) = u(1,t) = 0$, $0 < x < 1$, $t \geq 0$.

(Sol.)

$u(x,t) = X(x)T(t)$, $X(x)T'(t) = X''(x)T(t)$, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda$,

$X(0)=0=X(1) \Rightarrow \lambda = -(n\pi)^2$ and $X(x) = C_n \sin(n\pi x)$,

$\frac{T'(t)}{T(t)} = -(n\pi)^2$ and $T(0) = \text{constant} \Rightarrow T(t) = d_n e^{-n^2\pi^2 t}$,

$\therefore u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cdot e^{-n^2\pi^2 t}$ and $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$

$u(x,0) = 3\sin(2\pi x) \Rightarrow A_2 = 3$ but $A_n = 0$ for $n \neq 2 \Rightarrow u(x,t) = 3e^{-4\pi^2 t} \cdot \sin(2\pi x)$

6-3 Laplace Transform Solutions of Boundary Value Problems

Eg. Solve $\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} = y$, $\theta(x,0)=0$, and $\theta(0,y)=y$. [1990 台大化工研]

$$\text{(Sol.) } L[\theta(x, y)] = \int_0^{\infty} \theta(x, y)e^{-sy} dy = \Theta(x, s), \quad L[y] = \frac{1}{s^2} = L[\theta(0, y)] = \Theta(0, s)$$

$$\Rightarrow \frac{d\Theta(x, s)}{dx} + s\Theta(x, s) - \theta(x, 0) = \frac{1}{s^2} \Rightarrow \Theta(x, s) = A(s) \cdot e^{-sx} + \frac{1}{s^3}$$

$$\Theta(0, s) = A(s) + \frac{1}{s^3} = \frac{1}{s^2} \Rightarrow A(s) = \frac{1}{s^2} - \frac{1}{s^3} \Rightarrow \Theta(x, s) = \left(\frac{1}{s^2} - \frac{1}{s^3} \right) e^{-sx} + \frac{1}{s^3}$$

$$\Rightarrow \theta(x, y) = \left[y - x - \frac{1}{2}(y-x)^2 \right] \cdot u(y-x) + \frac{y^2}{2} = \begin{cases} \frac{y^2}{2}, & y < x \\ y - x - \frac{1}{2}(y-x)^2 + \frac{y^2}{2}, & y \geq x \end{cases}$$

Note: This partial differential equation can not be solved by separation of variables.

Eg. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t)=u(1,t)=u(x,0)=0$, and $\frac{\partial u}{\partial t}|_{t=0} = \sin(\pi x)$.

$$\text{(Sol.) } L[u(x, t)] = \int_0^{\infty} u(x, t)e^{-st} dt = U(x, s)$$

$$\frac{d^2 U(x, s)}{dx^2} = s^2 U(x, s) - su(x, 0) - \frac{\partial u}{\partial t}|_{t=0} = s^2 U(x, s) - \sin(\pi x)$$

$$\Rightarrow \frac{d^2 U(x, s)}{dx^2} - s^2 U(x, s) = -\sin(\pi x) \Rightarrow U(x, s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{\sin \pi x}{s^2 + \pi^2}$$

$$\begin{cases} u(0, t) = 0 \\ u(1, t) = 0 \end{cases} \Rightarrow \begin{cases} U(0, s) = 0 \\ U(1, s) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow u(x, t) = \frac{1}{\pi} \sin(\pi x) \cdot \sin(\pi t)$$

Eg. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0)=3\sin(2\pi x)$, $u(0,t)=u(1,t)=0$, $0 < x < 1, t \geq 0$.

$$\text{(Sol.) } L[u(x, t)] = \int_0^{\infty} u(x, t)e^{-st} dt = U(x, s) \Rightarrow sU(x, s) - u(x, 0) = \frac{d^2}{dx^2} U(x, s)$$

$$\frac{d^2}{dx^2} U(x, s) - sU(x, s) = -3\sin(2\pi x) \Rightarrow U(x, s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x} + \frac{3}{s + 4\pi^2} \cdot \sin(2\pi x)$$

$$L[u(0, t)] = U(0, s) = 0, \quad L[u(1, t)] = U(1, s) = 0 \Rightarrow c_1 = 0, \quad c_2 = 0$$

$$U(x, s) = \frac{3}{s + 4\pi^2} \cdot \sin(2\pi x) \Rightarrow u(x, t) = L^{-1} \left[\frac{3}{s + 4\pi^2} \cdot \sin(2\pi x) \right] = 3e^{-4\pi^2 t} \cdot \sin(2\pi x)$$

6-4 Fourier Transform Solutions of Boundary Value Problems

Error function: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

Complementary error function: $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$

Eg. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = e^{-x^2}$, $-\infty < x < \infty$, $t > 0$. [2011 成大電研、2001 台大電研類
似題]

$$\text{(Sol.) } \mathfrak{F}[u(x,t)] = U(\omega, t), \quad \mathfrak{F}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = -\omega^2 U(\omega, t)$$

$$\frac{d}{dt}U(\omega, t) = -\omega^2 U(\omega, t) \Rightarrow U(\omega, t) = A e^{-\omega^2 t}$$

$$\text{According to } \mathfrak{F}[e^{-a^2 x^2}] = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}},$$

$$U(\omega, 0) = A = \mathfrak{F}[u(x,0)] = \mathfrak{F}[e^{-x^2}] = \sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}} \Rightarrow A = \sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}}$$

$$\text{Let } b^2 = t + 1/4 \text{ and according to } \mathfrak{F}^{-1}[e^{-b^2 \omega^2}] = \frac{e^{-\frac{x^2}{4b^2}}}{2b\sqrt{\pi}},$$

$$\Rightarrow u(x,t) = \mathfrak{F}^{-1}[U(\omega, t)] = \mathfrak{F}^{-1}[\sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}} e^{-\omega^2 t}] = \sqrt{\pi} \cdot \mathfrak{F}^{-1}[e^{-(t+\frac{1}{4})\omega^2}] = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$$

Eg. Solve $\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = 4e^{-5|x|}$, $\frac{\partial u(x,0)}{\partial t} = 0$, $-\infty < x < \infty$, $t > 0$.

$$\text{(Sol.) } \mathfrak{F}[u(x,t)] = U(\omega, t), \quad \mathfrak{F}\left[9 \frac{\partial^2 u(x,t)}{\partial x^2}\right] = -9\omega^2 U(\omega, t)$$

$$\frac{d^2}{dt^2}U(\omega, t) = -9\omega^2 U(\omega, t) \Rightarrow U(\omega, t) = A \cos(3\omega t) + B \sin(3\omega t)$$

$$\frac{d}{dt}U(\omega, t) = -3\omega A \sin(3\omega t) + 3\omega B \cos(3\omega t)$$

$$\text{According to } \mathfrak{F}[e^{-a|x|}] = \frac{2a}{a^2 + \omega^2}, \quad U(\omega, 0) = A = \mathfrak{F}[u(x,0)] = \mathfrak{F}[4e^{-5|x|}] = \frac{40}{25 + \omega^2}$$

$$\mathfrak{F}\left[\frac{\partial u(x,0)}{\partial t}\right] = \frac{d}{dt}U(\omega, 0) = 0 = 3\omega B \Rightarrow B = 0, \quad \therefore U(\omega, t) = \frac{40}{\omega^2 + 25} \cdot \cos(3\omega t)$$

$$\text{By } \mathfrak{F}^{-1}[e^{ja\omega} F(\omega)] = f(x+a) \text{ and } \mathfrak{F}^{-1}[e^{-ja\omega} F(\omega)] = f(x-a),$$

$$u(x,t) = \mathfrak{F}^{-1}[U(\omega, t)] = \mathfrak{F}^{-1}\left[\frac{40}{\omega^2 + 25} \cdot \cos(3\omega t)\right] = \mathfrak{F}^{-1}\left[\frac{40}{\omega^2 + 25} \cdot \frac{e^{i3\omega t} + e^{-i3\omega t}}{2}\right]$$

$$= \mathfrak{F}^{-1}\left[\frac{20}{\omega^2 + 25} \cdot (e^{i3\omega t} + e^{-i3\omega t})\right] = 2\mathfrak{F}^{-1}\left[\frac{2 \cdot 5}{\omega^2 + 25} \cdot (e^{i3\omega t} + e^{-i3\omega t})\right] = 2e^{-5|x+3t|} + 2e^{-5|x-3t|}$$