Chapter 6 Partial Differential Equations (PDE)

6-1 Classification of Partial Differential Equations

The first-order linear PDE: $a(x,y)\dfrac{\partial u}{\partial x}+b(x,y)\dfrac{\partial u}{\partial y}+f(x,y)u+g(x,y)=0$

The second-order linear PDE: $a(x,y)\dfrac{\partial^2 u}{\partial x^2}+b(x,y)\dfrac{\partial^2 u}{\partial x\partial y}+c(x,y)\dfrac{\partial^2 u}{\partial y^2}$

$$+d(x,y)\frac{\partial u}{\partial x}+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u+g(x,y)=0$$

$$\begin{cases} hyperbolic\ at\ (x_0,y_0):\ \ \Delta(x_0,y_0)=b(x_0,y_0)^2-4a(x_0,y_0)c(x_0,y_0)>0 \\ elliptic\ at\ (x_0,y_0):\ \ \Delta(x_0,y_0)<0 \\ parabolic\ at\ (x_0,y_0):\ \ \Delta(x_0,y_0)=0 \end{cases}$$

Notations: $\dfrac{\partial^2 u}{\partial x^2}=u_{xx}$, $\dfrac{\partial^2 u}{\partial x\partial y}=u_{xy}$, $\dfrac{\partial u}{\partial x}=u_x$, etc.

Wave equation: $u_{tt}=a^2(u_{xx}+u_{yy}+u_{zz})+bu_t+F$

Heat equation or Diffusion equation: $u_t=a^2(u_{xx}+u_{yy}+u_{zz})=a^2\nabla^2 u$

Laplace's and Poisson's equations: $\nabla^2 u=u_{xx}+u_{yy}+u_{zz}=\begin{cases}0\\ \rho\end{cases}$

Schrodinger's equation: $i\hbar\dfrac{\partial\psi}{\partial t}=\dfrac{-\hbar^2}{2m}\nabla^2\psi+V\psi$ in quantum mechanics.

6-2 Separation-of-Variable Method

Eg. Solve $\dfrac{\partial^2\theta}{\partial x^2}=\dfrac{\partial^2\theta}{\partial t^2}$, $\theta(x,0)=x$, $\theta(0,t)=0$, $\dfrac{\partial\theta}{\partial t}\Big|_{t=0}=0$, **and** $\dfrac{\partial\theta}{\partial x}\Big|_{x=1}=0$.

(Sol.) Let $\theta(x,t)=X(x)T(t)$, $X''(x)T(t)=X(x)T''(t)$, $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{T(t)}=\lambda$

$$X(0)=0,\ X'(1)=0\Rightarrow -\lambda=\left[\frac{(2n-1)\pi}{2}\right]^2,\quad X_n=C_n\sin\left[\frac{(2n-1)\pi}{2}x\right]$$

$$\frac{T''(t)}{T(t)}=-\left[\frac{(2n-1)\pi}{2}\right]^2,\ T(0)=\text{constant},\ T'(0)=0\Rightarrow T_n=d_n\cos\left[\frac{(2n-1)\pi}{2}t\right],$$

$$\therefore\ \theta(x,t)=\sum_{n=1}^{\infty}A_n\cos\left[\frac{(2n-1)\pi}{2}t\right]\cdot\sin\left[\frac{(2n-1)\pi}{2}x\right]$$

$$\theta(x,0)=x=\sum_{n=1}^{\infty}A_n\sin\left[\frac{(2n-1)\pi}{2}x\right]\Rightarrow A_n=\frac{\int_{-1}^{1}x\sin\left[\frac{(2n-1)\pi}{2}x\right]dx}{\int_{-1}^{1}\sin^2\left[\frac{(2n-1)\pi}{2}x\right]dx}=\frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$, $u(0,t)=u(1,t)=u(x,0)=0$, and $\dfrac{\partial u}{\partial t}_{|t=0}=\sin(\pi x)$.

(Sol.) $u(x,t)=X(x)T(t)$, $X''(x)T(t)=X(x)T''(t)$, $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{T(t)}=\lambda$,

$X(0)=0=X(1)\Rightarrow\lambda=-(n\pi)^2$ and $X(x)=C_n\sin(n\pi x)$,

$\dfrac{T''(t)}{T(t)}=-(n\pi)^2$ and $T(0)=0$, $T'(0)=\text{constant}\Rightarrow T(t)=d_n\sin(n\pi t)$,

$\therefore\ u(x,t)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\cdot\sin(n\pi t)$, $\dfrac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty}n\pi A_n\sin(n\pi x)\cdot\cos(n\pi t)$

$\dfrac{\partial u}{\partial t}_{|t=0}=\sin(\pi x)\Rightarrow\pi A_1=1\Rightarrow A_1=1/\pi$ but $A_n=0$ for $n\neq1\Rightarrow u(x,t)=\dfrac{1}{\pi}\sin(\pi x)\cdot\sin(\pi t)$

Eg. Solve $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$, $u(x,0)=3\sin(2\pi x)$, $u(0,t)=u(1,t)=0$, $0<x<1$, $t\geq0$.

(Sol.)

$u(x,t)=X(x)T(t)$, $X(x)T'(t)=X''(x)T(t)$, $\dfrac{X''(x)}{X(x)}=\dfrac{T'(t)}{T(t)}=\lambda$,

$X(0)=0=X(1)\Rightarrow\lambda=-(n\pi)^2$ and $X(x)=C_n\sin(n\pi x)$,

$\dfrac{T'(t)}{T(t)}=-(n\pi)^2$ and $T(0)=\text{constant}\Rightarrow T(t)=d_n e^{-n^2\pi^2 t}$,

$\therefore\ u(x,t)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\cdot e^{-n^2\pi^2 t}$ and $u(x,0)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)$

$u(x,0)=3\sin(2\pi x)\Rightarrow A_2=3$ but $A_n=0$ for $n\neq2\Rightarrow u(x,t)=3e^{-4\pi^2 t}\cdot\sin(2\pi x)$

Eg. For $\dfrac{\partial^2 y}{\partial t^2}=\dfrac{\partial^2 y}{\partial x^2}$**,** $y(x,0)=f(x)=0.1\left(1-\left|\dfrac{2x}{\pi}-1\right|\right)=\begin{cases}\dfrac{0.2x}{\pi} & ,0<x<\dfrac{\pi}{2}\\ \left(1-\dfrac{x}{\pi}\right)0.2 & ,\dfrac{\pi}{2}<x<\pi\end{cases}$**, and** $y(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin(nx)$**.**

(a) Find the Fourier sine series for $f(x)$ on $[0,\pi]$.

(b) Find the ordinary differential equation and the initial condition for $b_n(t)$.

(c) Find $y(x,t)$. [中央電研]

(Sol.) (a) $f(x)=\sum_{n=1}^{\infty}a_n\sin\dfrac{n\pi x}{L}=\sum_{n=1}^{\infty}a_n\sin(nx)$, $2L=2\pi$, $L=\pi$, $\dfrac{n\pi x}{L}=nx$

$$a_n=\frac{1}{L}\int_{-\pi}^{\pi}f(x)\sin(nx)dx=\frac{2}{L}\int_0^{\pi}f(x)\sin(nx)dx$$

$$=\frac{2}{\pi}\left[\int_0^{\pi/2}\frac{0.2x}{\pi}\cdot\sin(nx)dx+\int_{\pi/2}^{\pi}0.2\left(1-\frac{x}{\pi}\right)\sin(nx)dx\right]=\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore\ f(x)=\sum_{n=1}^{\infty}\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cdot\sin(nx)$$

(b) $y(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin(nx)$, $y(x,0)=\sum_{n=1}^{\infty}b_n(0)\sin(nx)=f(x)$

$\Rightarrow\sum_{n=1}^{\infty}b_n''(t)\sin(nx)=\sum_{n=1}^{\infty}b_n(t)\cdot n^2\cdot(-\sin(nx))$

$\Rightarrow b_n''(t)+n^2b_n(t)=0$ is the ordinary differential equation.

$\Rightarrow b_n(t)=\alpha_n\cos(nt)+\beta_n\sin(nt)$

$y(x,0)=f(x)=\sum_{n=1}^{\infty}\dfrac{0.8}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)\cdot\sin(nx)=\sum_{n=1}^{\infty}b_n(0)\sin(nx)$

$\Rightarrow b_n(0)=\dfrac{0.8}{n^2\pi^2}\cdot\sin\left(\dfrac{n\pi}{2}\right)=\alpha_n$ (the initial condition), $\beta_n=0$

(c) $y(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin(nx)=\sum_{n=1}^{\infty}\alpha_n\cos(nt)\cdot\sin(nx)$

$$=\sum_{n=1}^{\infty}\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cos(nt)\cdot\sin(nx)$$

6-3 Laplace Transform Solutions of Boundary Value Problems

Eg. Solve $\frac{\partial \theta}{\partial x}+\frac{\partial \theta}{\partial y}=y$**,** $\theta(x,0)=0$**, and** $\theta(0,y)=y$**. [台大化工研]**

(Sol.) $L[\theta(x,y)]=\int_0^\infty \theta(x,y)e^{-sy}dy=\Theta(x,s)$, $L[y]=\frac{1}{s^2}=L[\theta(0,y)]=\Theta(0,s)$

$$\Rightarrow \frac{d\Theta(x,s)}{dx}+s\Theta(x,s)-\theta(x,0)=\frac{1}{s^2}\Rightarrow \Theta(x,s)=A(s)\cdot e^{-sx}+\frac{1}{s^3}$$

$$\Theta(0,s)=A(s)+\frac{1}{s^3}=\frac{1}{s^2}\Rightarrow A(s)=\frac{1}{s^2}-\frac{1}{s^3}\Rightarrow \Theta(x,s)=\left(\frac{1}{s^2}-\frac{1}{s^3}\right)e^{-sx}+\frac{1}{s^3}$$

$$\Rightarrow \theta(x,y)=\left[y-x-\frac{1}{2}(y-x)^2\right]\cdot u(y-x)+\frac{y^2}{2}=\begin{cases}\frac{y^2}{2}, \; y<x\\ y-x-\frac{1}{2}(y-x)^2+\frac{y^2}{2}, \;\; y\ge x\end{cases}$$

Note: This partial differential equation can not be solved by separation of variables.

Eg. Solve $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$**,** $u(0,t)=u(1,t)=u(x,0)=0$**, and** $\frac{\partial u}{\partial t}_{|t=0}=\sin(\pi x)$**.**

(Sol.) $L[u(x,t)]=\int_0^\infty u(x,t)e^{-st}dt=U(x,s)$

$$\frac{d^2U(x,s)}{dx^2}=s^2U(x,s)-su(x,0)-\frac{\partial u}{\partial t}_{|t=0}=s^2U(x,s)-\sin(\pi x)$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2}-s^2U(x,s)=-\sin(\pi x)\Rightarrow U(x,s)=c_1e^{sx}+c_2e^{-sx}+\frac{\sin \pi x}{s^2+\pi^2}$$

$$\begin{cases}u(0,t)=0\\ u(1,t)=0\end{cases}\Rightarrow \begin{cases}U(0,s)=0\\ U(1,s)=0\end{cases}\Rightarrow \begin{cases}c_1=0\\ c_2=0\end{cases}\Rightarrow u(x,t)=\frac{1}{\pi}\sin(\pi x)\cdot\sin(\pi t)$$

Eg. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$**,** $u(x,0)=3\sin(2\pi x)$**,** $u(0,t)=u(1,t)=0$**,** $0<x<1$**,** $t\ge 0$**.**

(Sol.) $L[u(x,t)]=\int_0^\infty u(x,t)e^{-st}dt=U(x,s)\Rightarrow sU(x,s)-u(x,0)=\frac{d^2}{dx^2}U(x,s)$

$$\frac{d^2}{dx^2}U(x,s)-sU(x,s)=-3\sin(2\pi x)\Rightarrow U(x,s)=c_1e^{-\sqrt{s}x}+c_2e^{\sqrt{s}x}+\frac{3}{s+4\pi^2}\cdot\sin(2\pi x)$$

$$L[u(0,t)]=U(0,s)=0,\;\; L[u(1,t)]=U(1,s)=0\Rightarrow c_1=0,\;\; c_2=0$$

$$U(x,s)=\frac{3}{s+4\pi^2}\cdot\sin(2\pi x)\Rightarrow u(x,t)=L^{-1}\left[\frac{3}{s+4\pi^2}\cdot\sin(2\pi x)\right]=3e^{-4\pi^2 t}\cdot\sin(2\pi x)$$

6-4 Fourier Transform Solutions of Boundary Value Problems

Eg. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$, $u(x,0)=e^{-x^2}$, $-\infty<x<\infty$, $t>0$. **[台大電研類似題]**

(Sol.) $\Im[u(x,t)]=U(\omega,t)$, $\Im\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]=-\omega^2 U(\omega,t)$

$$\frac{d}{dt}U(\omega,t)=-\omega^2 U(\omega,t) \Rightarrow U(\omega,t)=Ae^{-\omega^2 t}$$

According to $\Im[e^{-a^2x^2}]=\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$,

$$U(\omega,0)=A=\Im[u(x,0)]=\Im[e^{-x^2}]=\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}} \Rightarrow A=\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}$$

Let $b^2=t+1/4$ and according to $\Im^{-1}[e^{-b^2\omega^2}]=\frac{e^{-\frac{x^2}{4b^2}}}{2b\sqrt{\pi}}$,

$$\Rightarrow u(x,t)=\Im^{-1}[U(\omega,t)]=\Im^{-1}[\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}e^{-\omega^2 t}]=\sqrt{\pi}\cdot\Im^{-1}[e^{-(t+\frac{1}{4})\omega^2}]=\frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$$

Error function: $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$

Complementary error function: $erfc(x)=1-erf(x)$