Chapter 7 Vector Analysis

7-1 Vector Functions

One-variable vector function: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Multi-variable vector function: $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$

The *derivatives* of vector functions: $\dfrac{d\vec{R}(t)}{df(t)} = \dfrac{d\vec{R}(t)}{dt}\cdot\dfrac{dt}{df(t)} = \dfrac{d\vec{R}(t)}{dt} / \dfrac{df(t)}{dt}$

$$\frac{\partial \vec{F}(x,y,z)}{\partial g(x,y,z)} = \frac{\partial \vec{F}(x,y,z)}{\partial x}\cdot\frac{\partial x}{\partial g(x,y,z)} + \frac{\partial \vec{F}(x,y,z)}{\partial y}\cdot\frac{\partial y}{\partial g(x,y,z)} + \frac{\partial \vec{F}(x,y,z)}{\partial z}\cdot\frac{\partial z}{\partial g(x,y,z)}$$

Eg. For $\vec{R}(t) = 2t\hat{i} - \cos(3t)\hat{j} + t^3\hat{k}$ **,** $0 \le t \le 1$**, find** $\vec{R}'(t) = d\vec{R}(t)/dt$ **. Let** $s(t) = \int_0^1 \sqrt{4 + 9\sin^2(3t) + 9t^4}\cdot dt$**, then find** $d\vec{R}(t)/ds(t)$**.**

(Sol) $d\vec{R}(t)/dt = 2\hat{i} + 3\sin(3t)\hat{j} + 3t^2\hat{k}$

$$\therefore\ \frac{d\vec{R}(t)}{ds(t)} = \frac{d\vec{R}(t)}{dt}\cdot\frac{dt}{ds(t)} = \frac{2\hat{i} + 3\sin(3t)\hat{j} + 3t^2\hat{k}}{\sqrt{4 + 9\sin^2(3t) + 9t^4}}$$

Some theorems of derivatives of vector functions:

1. $(\vec{F}\cdot\vec{G})' = \vec{F}'\cdot\vec{G} + \vec{F}\cdot\vec{G}'$

2. $(\vec{F}\times\vec{G})' = \vec{F}'\times\vec{G} + \vec{F}\times\vec{G}'$

3. $(\vec{F}\times\vec{F}')' = \vec{F}\times\vec{F}''$

(Proof) $(\vec{F}\times\vec{F}')' = \vec{F}'\times\vec{F}' + \vec{F}\times\vec{F}'' = \vec{F}\times\vec{F}''$

4. For $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ **, if** $\vec{R}(t)$ **does not change direction, then** $\vec{R}(t)\times\vec{R}'(t) = 0$**, and vice versa.**

5. Let $\vec{R}(t)$ **denote the position of a particle at time *t*. If the particle moves so that equal areas are swept out in equal times, then we have** $\vec{R}(t)\times\vec{R}''(t) = 0$**, and vice versa. (Kepler's law)**

(Proof)

$$\text{Area} = \frac{1}{2}R^2\theta \quad \text{and} \quad |\vec{R}(t+\Delta t) - \vec{R}(t)| \approx R\theta$$

$$\therefore\ 2\ \text{area} = R^2\theta = \vec{R}(t)\times[\vec{R}(t+\Delta t) - \vec{R}(t)]$$

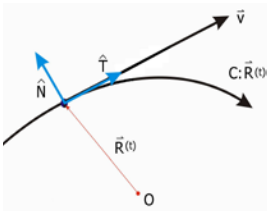
$$\text{If area} = 0 \Leftrightarrow \vec{R}(t)\times[\vec{R}(t+\Delta t) - \vec{R}(t)] = 0 \Leftrightarrow \vec{R}(t)\times\vec{R}'(t)\cdot\Delta t = 0$$

$$\Leftrightarrow \vec{R}(t)\times\vec{R}'(t) = 0$$

Equal area in equal time

$$\Leftrightarrow \vec{R}(t)\times\vec{R}'(t) = \text{constant} \Leftrightarrow [\vec{R}(t)\times\vec{R}'(t)]' = 0 \Leftrightarrow \vec{R}(t)\times\vec{R}''(t) = 0$$

7-2 Differential Geometry



Position vector: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Velocity: $\vec{v}(t) = \vec{R}'(t) = \dfrac{dx(t)}{dt}\hat{i} + \dfrac{dy(t)}{dt}\hat{j} + \dfrac{dz(t)}{dt}\hat{k}$

Arc length: $s(t) = \int_{t_1}^{t_2} |\vec{R}'(t)|\,dt,\ \dfrac{ds}{dt} = |\vec{R}'(t)| = |\vec{v}(t)|$

Acceleration: $\vec{a}(t) = \vec{v}'(t) = \dfrac{d^2x(t)}{dt^2}\hat{i} + \dfrac{d^2y(t)}{dt^2}\hat{j} + \dfrac{d^2z(t)}{dt^2}\hat{k}$

Curvature: $\kappa = \left|\dfrac{d\hat{T}}{ds}\right|$, where $\hat{T} = \dfrac{\vec{R}'(t)}{|\vec{R}'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{d\vec{R}(t)}{ds(t)}$

Eg. *C*: $\vec{R}(t) = t\hat{i} + (t-2)\hat{j} + (3t-1)\hat{k}$ **is a straight line.**

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{11}},\quad \kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}\right| = 0.$$

Eg. C: $\vec{R}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + 4\hat{k}$ **is a circle of radius 2 at *z*=4.**

$$\hat{T} = \frac{-2\sin(t)\hat{i} + 2\cos(t)\hat{j}}{2},\quad \kappa = \left|\frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}\right| = \frac{1}{2} = \frac{1}{r}.$$

Theorem $\vec{a} = \dfrac{d|\vec{v}|}{dt}\hat{T} + \dfrac{|\vec{v}|^2}{\rho}\hat{N}$ = **tangential acceleration+centripetal acceleration**

(Proof) $\vec{a}(t) = \dfrac{d\vec{v}(t)}{dt} = \dfrac{d}{dt}[|\vec{v}(t)|\hat{T}] = \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|\cdot\dfrac{d\hat{T}}{dt}$

$= \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|\cdot\left(\dfrac{ds}{dt}\cdot\dfrac{d\hat{T}}{ds}\right) = \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|^2\dfrac{d\hat{T}}{ds}$. Define $\hat{N} = \rho\dfrac{d\hat{T}}{ds}$

$\Rightarrow \vec{a}(t) = \dfrac{d|\vec{v}(t)|}{dt}\hat{T} + \dfrac{|\vec{v}(t)|^2}{\rho}\hat{N}$ $\quad(\hat{N}\perp\hat{T} \Leftarrow \hat{T}\cdot\hat{T} = 1)$, $\dfrac{d(\hat{T}\cdot\hat{T})}{ds} = 0$

Eg. For $\vec{R}(t) = [\cos(t) + t\sin(t)]\hat{i} + [\sin(t) - t\cos(t)]\hat{j} + t^2\hat{k}$**, *t*>0, we have**

$$\vec{v}(t) = t\cos(t)\hat{i} + t\sin(t)\hat{j} + 2t\hat{k}$$

$$\vec{a}(t) = [\cos(t) - t\sin(t)]\hat{i} + [\sin(t) + t\cos(t)]\hat{j} + 2\hat{k}$$

$$\hat{T} = \frac{\vec{v}(t)}{|v(t)|} = \frac{1}{\sqrt{5}}\cos(t)\hat{i} + \frac{1}{\sqrt{5}}\sin(t)\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$$

$$\rho = \frac{1}{\kappa} = \frac{1}{\left|\frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}\right|} = \frac{1}{\left|\frac{d\hat{T}}{dt}\cdot\frac{1}{|\vec{v}(t)|}\right|} = 5t$$

$$\hat{N} = \rho\frac{d\hat{T}}{ds} = \rho\frac{dt}{ds}\cdot\frac{d\hat{T}}{dt} = \frac{\rho}{|\vec{v}(t)|}\cdot\frac{d\hat{T}}{dt} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$

$\therefore\ \vec{a}(t) = \sqrt{5}\hat{T} + t\hat{N}$, $a_t = \sqrt{5}$, $a_n = t$

Binormal vector: $\hat{B} = \hat{T} \times \hat{N}$

Fernet formulae:
$$\begin{cases} \dfrac{d\hat{T}}{ds} = \kappa\hat{N} = \dfrac{\hat{N}}{\rho} & (1) \\ \dfrac{d\hat{N}}{ds} = -\kappa\hat{T} + \tau\hat{B} & (2) \\ \dfrac{d\hat{B}}{ds} = -\tau\hat{N} & (3) \end{cases}$$

***Torsion* of a curve: τ**

(Proof of (3)) $\dfrac{d\hat{B}}{ds} = \dfrac{d}{ds}(\hat{T}\times\hat{N}) = \dfrac{d\hat{T}}{ds}\times\hat{N} + \hat{T}\times\dfrac{d\hat{N}}{ds} = \hat{T}\times(-\kappa\hat{T}) + \hat{T}\times(\tau\hat{B}) = -\tau\hat{N}$

Note: $\hat{T}$, $\hat{N}$, *and* $\hat{B}$ are unit vectors.

Basic theorems of curvature and torsion:

$\kappa = \dfrac{|\vec{R}'\times\vec{R}''|}{|\vec{R}'|^3}$, $\tau = [\hat{T}, \hat{N}, \hat{N}'] = \dfrac{1}{\kappa^2}[\vec{R}', \vec{R}'', \vec{R}''']$, **where** $[\vec{A}, \vec{B}, \vec{C}] \equiv \vec{A}\cdot(\vec{B}\times\vec{C})$

Eg. For $C: \vec{R}(t) = 3\cos(t)\hat{i} + 3\sin(t)\hat{j} + 4t\hat{k}$, **find** $\hat{T}$, $\hat{N}$, $\hat{B}$, ***κ*, *τ*, and *ρ*.**

(Sol.) $\hat{T} = \dfrac{\vec{v}}{|\vec{v}|} = -\dfrac{3}{5}\sin(t)\hat{i} + \dfrac{3}{5}\cos(t)\hat{j} + \dfrac{4}{5}\hat{k}$, $\kappa = \dfrac{|\vec{R}'\times\vec{R}''|}{|R'|^3} = \dfrac{3}{25}$, $\rho = \dfrac{1}{\kappa} = \dfrac{25}{3}$

$\hat{N} = \rho\dfrac{d\hat{T}}{ds} = \dfrac{\rho}{|\vec{v}(t)|}\cdot\dfrac{d\hat{T}}{dt} = -\cos(t)\hat{i} - \sin(t)\hat{j}$, $\hat{B} = \hat{T}\times\hat{N} = \dfrac{4}{5}\sin(t)\hat{i} - \dfrac{4}{5}\cos(t)\hat{j} + \dfrac{3}{5}\hat{k}$

$\dfrac{d\hat{B}}{ds} = \dfrac{1}{|\vec{v}(t)|}\dfrac{d\hat{B}}{dt} = \dfrac{4}{25}\cos(t)\hat{i} + \dfrac{4}{25}\sin(t)\hat{j} = -\tau\hat{N} = -\tau(-\cos(t)\hat{i} - \sin(t)\hat{j})$

$\Rightarrow \tau = \dfrac{4}{25}$

Eg. Consider the curve: $\hat{r} = a\cos(t)\hat{i} + a\sin(t)\hat{j} + bt\hat{k}$, $0 \le t \le 2\pi$. **What is the equation of tangential vector at *t*=*π*/2. 【中山機研】**

(Sol.) $\vec{v} = \dfrac{d\vec{r}}{dt} = -a\sin(t)\hat{i} + a\cos(t)\hat{j} + b\hat{k}$. At $t = \dfrac{\pi}{2} \Rightarrow \vec{v} = -a\hat{i} + b\hat{k}$

$\hat{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{-a\hat{i} + b\hat{k}}{\sqrt{a^2 + b^2}}$

7-3 Gradient, Divergence, and Curl in the Rectangular Coordinate System

Gradient $\nabla\phi(x,y,z)$ **:** $\nabla\phi(x,y,z)=\frac{\partial\phi(x,y,z)}{\partial x}\hat{i}+\frac{\partial\phi(x,y,z)}{\partial y}\hat{j}+\frac{\partial\phi(x,y,z)}{\partial z}\hat{k}$ is a vector function if $\phi(x,y,z)$ is a scalar function.

Divergence $\nabla\cdot\vec{F}(x,y,z)$ **:** $\nabla\cdot\vec{F}(x,y,z)=\frac{\partial F_1(x,y,z)}{\partial x}+\frac{\partial F_2(x,y,z)}{\partial y}+\frac{\partial F_3(x,y,z)}{\partial z}$ is a scalar function if $\vec{F}(x,y,z)=F_1(x,y,z)\hat{i}+F_2(x,y,z)\hat{j}+F_3(x,y,z)\hat{k}$ is a vector function.

Curl $\nabla\times\vec{F}(x,y,z)$**:**

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\hat{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\hat{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\hat{k}$$

is a vector function if $\vec{F}(x,y,z)=F_1(x,y,z,)\hat{i}+F_2(x,y,z)\hat{j}+F_3(x,y,z)\hat{k}$ is a vector function.

Eg. $\vec{F}=x\hat{i}+y\hat{j}+z\hat{k}$**, compute** $\nabla\cdot\vec{F}$ **and** $\nabla\times\vec{F}$**.**

(Sol.) $\nabla\cdot\vec{F}=1+1+1=3$, $\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}=0$

Eg. $\vec{F}(x,y,z)=x^2\hat{i}-2x^2y\hat{j}+2yz^4\hat{k}$**, find** $\nabla\times\vec{F}$ **and** $\nabla\cdot\vec{F}$ **at (1,-1,1).【中山電研】**

(Sol.) $\nabla\cdot\vec{F}=2x-2x^2+8yz^3,\ at\ (1,-1,1)\Rightarrow\nabla\cdot\vec{F}=-8$

$\nabla\times\vec{F}=2z^4\hat{i}-4xy\hat{k},\ at\ (1,-1,1)\Rightarrow\nabla\times\vec{F}=2\hat{i}+4\hat{k}$

Laplacian operator: $\nabla^2\phi=\nabla\cdot\nabla\phi=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})\phi$

Theorems **(*a*)** $\nabla\times(\nabla\times\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$

(*b*) $\nabla\cdot(\nabla\times\vec{F})=0,\ \forall\vec{F}\in C^2$

(*c*) $\nabla\times(\nabla\phi)=0,\ \forall\phi\in C^2$

(*d*) $\nabla\times(\vec{F}\times\vec{G})=(\vec{G}\cdot\nabla)\vec{F}-(\vec{F}\cdot\nabla)\vec{G}+(\nabla\cdot\vec{G})\vec{F}-(\nabla\cdot\vec{F})\vec{G}$

(*e*) $\nabla(\vec{F}\cdot\vec{G})=(\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$

(*f*) $\nabla\cdot(\vec{F}\times\vec{G})=\vec{G}\cdot(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G})$

(*g*) $\nabla\cdot(\phi\vec{F})=(\nabla\phi)\cdot\vec{F}+\phi(\nabla\cdot\vec{F})$

Theorem $\nabla\phi(x,y,z)\perp$ **the surface of $\phi(x,y,z)$=constant.**

(Proof) $\because$ $\phi(x,y,z)$=constant

$$\therefore\ d\,\phi(x,y,z)=0=\frac{\partial\phi(x,y,z)}{\partial x}dx+\frac{\partial\phi(x,y,z)}{\partial y}dy+\frac{\partial\phi(x,y,z)}{\partial z}dz$$

$$=\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})=\nabla\phi\cdot d\vec{R}$$

$\therefore$ $\nabla\phi(x,y,z)\perp d\vec{R}$, and $d\vec{R}$ is the tangential increment on the surface $\phi(x,y,z)$

Eg. Find the tangential plane and normal line to $z=x^2+y^2$ at (2,-2,8).

(Sol.) Let $\phi(x,y,z)=z-x^2-y^2$, $\nabla\phi=-2x\hat{i}-2y\hat{j}+\hat{k}$, and (2,-2,8) is on the surface.

For $z-x^2-y^2=0$, the normal vector at (2,-2,8) is $-4\hat{i}+4\hat{j}+\hat{k}$.

The tangential plane at (2,-2,8) is $-4(x-2)+4(y+2)+(z-8)=0\Rightarrow -4x+4y+z=-8$

The normal line is $\dfrac{x-2}{-4}=\dfrac{y+2}{4}=\dfrac{z-8}{1}$.

Eg. Find a unit normal vector of $z^2=4(x^2+y^2)$ at (1,0,2). 【中山電研】

Theorem $\nabla\cdot\vec{F}(x,y,z)$ **is the outward flux per unit volume of the flow at point (x,y,z) and time t.**

(Proof) $\vec{F}\ at\ P=F_x\hat{i}+F_y\hat{j}+F_z\hat{k}$

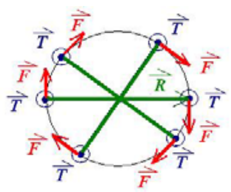
The x-direction flux $=\left(F_x+\dfrac{1}{2}\dfrac{\partial F_x}{\partial x}\Delta x\right)\Delta y\,\Delta z-\left(F_x-\dfrac{1}{2}\dfrac{\partial F_x}{\partial x}\Delta x\right)\Delta y\,\Delta z=\dfrac{\partial F_x}{\partial x}\Delta x\,\Delta y\,\Delta z$

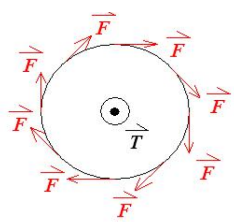
The y-direction flux $=\dfrac{\partial F_y}{\partial y}\Delta x\,\Delta y\,\Delta z$, and the z- direction flux $=\dfrac{\partial F_z}{\partial z}\Delta x\,\Delta y\,\Delta z$

$$\therefore\ \text{Total flux}=\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)\cdot\Delta x\,\Delta y\,\Delta z/\Delta x\,\Delta y\,\Delta z=\nabla\cdot\vec{F}$$

Theorem If $\vec{T}=\vec{F}\times\vec{R}$ and $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$, then $\vec{F}=\dfrac{1}{2}\nabla\times\vec{T}$.

(Proof) $\nabla\times\vec{T}=\nabla\times(\vec{F}\times\vec{R})=\nabla\times\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\F_1&F_2&F_3\\x&y&z\end{vmatrix}$





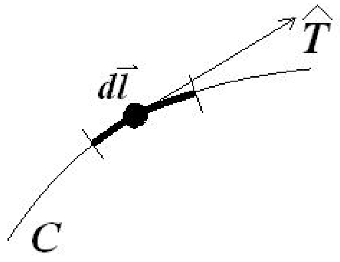
$$=\nabla\times[(F_2z-F_3y)\hat{i}+(F_3x-F_1z)\hat{j}+(F_1y-F_2x)\hat{k}]$$

$$=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\F_2z-F_3y&F_3x-F_1z&F_1y-F_2x\end{vmatrix}=2(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})=2\vec{F}\Rightarrow\vec{F}=\frac{1}{2}\nabla\times\vec{T}$$

7-4 Line Integrals & Surface Integrals

Line integral: $\exists$ Curve C: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},\ \ a \le t \le b$

$\exists$ Vector $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$



(a) $\int_C \vec{F}\cdot d\vec{l} = \int_a^b \vec{F}[x(t),y(t),z(t)]\cdot \vec{R}'(t)dt$,

(b) $\int_C \vec{F}\times d\vec{l} = \int_a^b \vec{F}[x(t),y(t),z(t)]\times \vec{R}'(t)dt$,

(c) $\int_C f(x,y,z)dl = \int_a^b f[x(t),y(t),z(t)]\cdot|\vec{R}'(t)|\,dt$

Eg. For $C:\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k}$ **,** $0 \le t \le 2\pi$**, and** $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$**, determine** $\int_C \vec{F}\cdot d\vec{l}$ **and** $\int_C \vec{F}\times d\vec{l}$ **.**

(Sol.) $C:\vec{R}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k},\ \ \vec{R}'(t) = -\sin(t)\hat{i} + \cos(t)\hat{j}$

$$\vec{F}(x(t),y(t),z(t)) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k}$$

$$\vec{F}\cdot d\vec{l} = \vec{F}(x(t),y(t),z(t))\cdot\vec{R}'(t)dt = (\cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k})\cdot(-\sin(t)\hat{i} + \cos(t)\hat{j})dt$$
$$= (-\cos(t)\sin(t) + \sin(t)\cos(t))dt = 0$$

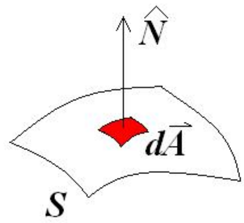
$$\vec{F}\times d\vec{l} = \vec{F}(x(t),y(t),z(t))\times\vec{R}'(t)dt = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(t) & \sin(t) & 1 \\ -\sin(t) & \cos(t) & 0 \end{vmatrix} dt$$

$$= [-\cos(t)\hat{i} - \sin(t)\hat{j} + \hat{k}]dt$$

$$\therefore\ \int_C \vec{F}\cdot d\vec{l} = 0,\ \ \int_C \vec{F}\times d\vec{l} = \int_0^{2\pi}[-\cos(t)\hat{i} - \sin(t)\hat{j} + \hat{k}]dt = 2\pi\hat{k}$$

Surface integral: $\exists$ Surface $S:\ z = \varphi(x,y)$. If $z = \varphi(x,y)$ has a projection on the xy-plane, then $\vec{N} = = -\frac{\partial\varphi}{\partial x}\hat{i} - \frac{\partial\varphi}{\partial y}\hat{j} + \hat{k}$ is a normal vector on S, and $\hat{N} = \vec{N}/|\vec{N}|$.

$\exists$ Vector $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$



(a) $\iint_S \vec{F}\cdot d\vec{A} = \iint_D \vec{F}(x,y,z)\cdot\hat{N}\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$,

(b) $\iint_S \vec{F}\times d\vec{A} = \iint_D \vec{F}(x,y,z)\times\hat{N}\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$,

(c) $\iint_S f(x,y,z)dA = \iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$,

Note: $d\vec{l} = \hat{T}dl$ is parallel to the tangential direction of the curve, but $d\vec{A} = \hat{N}dA$ is normal to the surface.

Eg. For $S: x^2+y^2+z^2=1, z\ge 0$ **, and** $\vec{F}(x,y,z)=x\hat{i}+y\hat{j}+z\hat{k}$ **, determine** $\iint_S \vec{F}\cdot d\vec{A}$ **and** $\iint_S \vec{F}\times d\vec{A}$**.**

(Sol.) $S:\ x^2+y^2+z^2=1 \wedge z\ge 0 \Rightarrow z=\sqrt{1-x^2-y^2}$, $\dfrac{\partial z}{\partial x}=\dfrac{-x}{\sqrt{1-x^2-y^2}}=-\dfrac{x}{z}$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{1-x^2-y^2}}=-\frac{y}{z}\Rightarrow \vec{N}=-\frac{\partial z}{\partial x}\hat{i}-\frac{\partial z}{\partial y}\hat{j}+\hat{k}=\frac{x}{z}\hat{i}+\frac{y}{z}\hat{j}+\hat{k},$$

$$\hat{N}=\frac{\vec{N}}{|\vec{N}|}=x\hat{i}+y\hat{j}+z\hat{k},\quad \vec{F}(x,y,z)=x\hat{i}+y\hat{j}+z\hat{k}\text{, and }\ x^2+y^2+z^2=1$$

$$\Rightarrow \vec{F}\cdot d\vec{A}=\vec{F}\cdot\hat{N}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy=\frac{x^2+y^2+z^2}{z}dxdy=\frac{dxdy}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \vec{F}\times d\vec{A}=\vec{F}\times\hat{N}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\x&y&z\end{vmatrix}\frac{dxdy}{z}=0$$

$$\therefore\ \iint_S \vec{F}\cdot d\vec{A}=\iint_{x^2+y^2\le 1}\frac{dxdy}{\sqrt{1-x^2-y^2}}=\int_0^{2\pi}\int_0^1\frac{rdrd\theta}{\sqrt{1-r^2}}=2\pi,\quad \iint_S \vec{F}\times d\vec{A}=0$$

Eg. $f=\dfrac{1}{r},\ r=\sqrt{x^2+y^2+z^2}$ **, find** $\iint_S \nabla f\cdot\hat{n}dA$ **for** S**:** $x^2+y^2+z^2=a^2$**.【中山電研】**

(Sol.) $S:\ x^2+y^2+z^2=a^2\Rightarrow z=\pm\sqrt{a^2-x^2-y^2}$,

For $z=\sqrt{a^2-x^2-y^2}\Rightarrow \vec{N}=-\dfrac{\partial z}{\partial x}\hat{i}-\dfrac{\partial z}{\partial y}\hat{j}+\hat{k}=\dfrac{x}{z}\hat{i}+\dfrac{y}{z}\hat{j}+\hat{k}$,

$$\hat{N}=\frac{\vec{N}}{|\vec{N}|}=\frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k}),\quad \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\frac{a}{z}$$

$$\nabla f=\nabla(\frac{1}{r})=\nabla(\frac{1}{\sqrt{x^2+y^2+z^2}})=\frac{-1}{(\sqrt{x^2+y^2+z^2})^3}(x\hat{i}+y\hat{j}+z\hat{k})=\frac{-1}{a^3}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow \nabla(\frac{1}{r})\cdot\hat{n}dA=\nabla(\frac{1}{r})\cdot\hat{N}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy=-\frac{x^2+y^2+z^2}{a^3z}dxdy=\frac{-dxdy}{a\cdot\sqrt{a^2-x^2-y^2}}$$

$$\iint_{z=\sqrt{a^2-x^2-y^2}}\nabla f\cdot\hat{n}dA=\iint_{x^2+y^2\le a^2}\frac{-dxdy}{a\cdot\sqrt{a^2-x^2-y^2}}=\frac{-1}{a}\int_0^{2\pi}\int_0^a\frac{rdrd\theta}{\sqrt{a^2-r^2}}=\frac{-2\pi}{a}\int_0^a\frac{rdr}{\sqrt{a^2-r^2}}$$

$=-2\pi$. Similarly, for $z=-\sqrt{a^2-x^2-y^2}$, $\iint_{z=-\sqrt{a^2-x^2-y^2}}\nabla f\cdot\hat{n}dA=-2\pi$

$$\therefore\ \iint_S\nabla f\cdot\hat{n}dA=(-2\pi)+(-2\pi)=-4\pi$$

Eg. **Compute the surface integral** $\iint_S (x+y+z)dA$ **, where** $S: z=x+y, 0\le y\le x, 0\le x\le 1$.【台大農工研】

(Sol.) $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{3}$, $\iint_S (x+y+z)dA=\int_0^1\int_0^x 2(x+y)\cdot\sqrt{3}dydx=\sqrt{3}$

Eg. **For** $\vec{F}=e^x\sin y\hat{i}+e^x\cos y\hat{j}+z^2\hat{k}$ **, find** $\int_C \vec{F}\cdot d\vec{r}$ **, where** $C:\ \vec{r}(t)=t^{1/2}\hat{i}+t^3\hat{j}+e^{\sqrt{t}}\hat{k}, 0\le t\le 1$. 【清大核工】 (Ans.) $e\cdot\sin(1)+\frac{e^3}{3}+\frac{2}{3}$

Eg. $\vec{F}=\hat{i}xy+\hat{j}yz+\hat{k}xz$**, evaluate** $\iint_S \vec{F}\cdot\hat{n}dA$ **for** $S: 0\le x\le 1, 0\le y\le 1, 0\le z\le 1$.
【中山電研】

Green's theorem **Let *C* be a regular, closed, positively-oriented curve enclosing a region *D*,** $\vec{F}(x,y)=F_1(x,y)\hat{i}+F_2(x,y)\hat{j}$.

$$\oint_C F_1(x,y)dx+F_2(x,y)dy=\iint_D\left[\frac{\partial F_2(x,y)}{\partial x}-\frac{\partial F_1(x,y)}{\partial y}\right]dxdy$$

Eg. $\hat{F}=\frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$**, find** $\oint_C \vec{F}\cdot d\vec{r}$**, where *C* is any closed curve.** 【成大電研】

(Sol.) $\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=0, \forall(x,y)\ne 0$, $\oint_C \vec{F}\cdot d\vec{r}=0$ if *C* does not enclose 0.

Else, $\oint \vec{F}\cdot d\vec{r}=2\pi$

Stokes's theorem **Let *S* be a regular surface with coherently oriented boundary *C*,** $\iint_S(\nabla\times\vec{F})\cdot d\vec{A}=\oint_C \vec{F}\cdot d\vec{\ell}$.

Divergence theorem **Let *S* be a regular, positive-oriented closed surface, enclosing a region *V*,** $\oiint_S \vec{F}\cdot d\vec{A}=\iiint_V(\nabla\cdot\vec{F})dxdydz$.

Eg. **Compute** $\oiint_S \vec{F}\cdot d\vec{A}$ **, where** $\vec{F}=(y^2+z^2)^{\frac{2}{3}}\hat{i}+\sin(x^2+z)\hat{j}+e^{x^2-y^2}\hat{k}$ **,** $S:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. 【成大工程科學所】

(Sol.) $\nabla\cdot\vec{F}=0\Rightarrow\oiint_S \vec{F}\cdot d\vec{A}=\iiint_V(\nabla\cdot\vec{F})dxdydz=0$

Eg. $f=\frac{1}{r},\ r=\sqrt{x^2+y^2+z^2}$ **, find** $\iint_S \nabla f\cdot\hat{n}dA$ **for** ***S*****:** $x^2+y^2+z^2=a^2$**.【中山電研】**

(Sol.)

$$\vec{F}=\nabla f=\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=\frac{-1}{(\sqrt{x^2+y^2+z^2})^3}(x\hat{i}+y\hat{j}+z\hat{k})=\frac{-1}{a^3}(x\hat{i}+y\hat{j}+z\hat{k}),$$

$$\nabla\cdot\vec{F}=\frac{-3}{a^3},\quad \iint_S \nabla f\cdot\hat{n}dA=\oiint_S \vec{F}\cdot d\vec{A}=\iiint_V(\nabla\cdot\vec{F})dxdydz=\frac{-3}{a^3}\cdot\frac{4a^3\pi}{3}=-4\pi$$

Green's identities

$\phi,\ \psi$ are scalars,

$$\begin{cases}(a)\iiint_V[\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi]dxdydz=\iint_S(\phi\nabla\psi)\cdot d\vec{A}\\ (b)\iiint_V[\phi\nabla^2\psi-\psi\nabla^2\phi]dxdydz=\iint_S[\phi\nabla\psi-\psi\nabla\phi]d\vec{A}\end{cases}.$$

7-5 Potential Theory

Potential: ϕ is called a potential for the vector field $\vec{F}$ if $\vec{F}=\nabla\phi$ or $\vec{F}=-\nabla\phi$

Test for a potential: If $\vec{F}$ and $\nabla\cdot\vec{F}$ are continuous in a simply-connected domain Ω, then $\vec{F}$ has a potential function. $\Leftrightarrow\nabla\times\vec{F}=0$

(Proof) 〝$\Rightarrow$〞: $\vec{F}=\pm\nabla\phi\Rightarrow\nabla\times\vec{F}=0$.

〝$\Leftarrow$〞: $\nabla\times\vec{F}=0$, $\oint_C\vec{F}\cdot d\vec{\ell}=\iint_S(\nabla\times\vec{F})\cdot d\vec{A}=0$

Let C be the boundary of ϕ *(x,y,z)*=constant, then we choose $\vec{F}=\pm\nabla\phi$, then $\int_C\vec{F}\cdot d\vec{\ell}=0$ is always valid. $\Rightarrow\vec{F}$ has a potential function.

Eg. Check (a) $\vec{F}=2xy\hat{i}+z^2\hat{j}+(x-y+z)\hat{k}$ **,**

and (b) $\vec{F}=(yze^{xyz}-4x)\hat{i}+(xze^{xyz}+z)\hat{j}+(xye^{xyz}+y)\hat{k}$ **, which does have a potential?**

(Sol.) (a) $\nabla\times\vec{F}=(-2z-1)\hat{i}-\hat{j}-2x\hat{k}\neq0$, $\therefore$ no potential.

(b) $\nabla\times\vec{F}=0$, $\therefore$ there exists a potential.

$$\vec{F}=\nabla\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}=(yze^{xyz}-4x)\hat{i}+(xze^{xyz}+z)\hat{j}+(xye^{xyz}+y)\hat{k}$$

$$\Rightarrow\phi(x,y,z)=e^{xyz}-2x^2+I(y,z)=e^{xyz}+yz+J(x,z)=e^{xyz}+yz+K(x,y)$$

$$\Rightarrow\phi(x,y,z)=e^{xyz}-2x^2+zy+\text{Constant}$$

Eg. $\vec{F}=(x^2+y^2+z^2)^n(x\hat{i}+y\hat{j}+z\hat{k})$ **, find a scalar potential** ϕ ***(x,y,z)*** **so that** $\vec{F}=-\nabla\phi$**.【台大材研】** (Ans.) $\phi(x,y,z)=\frac{-(x^2+y^2+z^2)^{n+1}}{2(n+1)}+C$

Theorem If $\vec{F}$ has a potential, then the line integral of $\vec{F}$ is independent of path in Ω. That is, $\int_C \vec{F}\cdot d\vec{\ell} = \int_K \vec{F}\cdot d\vec{\ell}$, whenever *C* and *K* are regular curves in Ω with the same initial point and the same terminal point.

(Proof) $\int_C \vec{F}\cdot d\vec{\ell} - \int_K \vec{F}\cdot d\vec{\ell} = \oint_{C'} \vec{F}\cdot d\vec{\ell}$

If $\vec{F}$ has a potential, then $\oint_{C'} \vec{F}\cdot d\vec{\ell} = 0$, ∴ $\int_C \vec{F}\cdot d\vec{\ell} = \int_K \vec{F}\cdot d\vec{\ell}$.

Theorem Let $\vec{F}$ be a 2-D vector field of a simply-connected domain Ω. Then $\vec{F}$ has a potential on Ω. ⇔ $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$. (In this case, $\vec{F} = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$)

(Proof) If $\vec{F} = \nabla\phi$, then $F_1 = \frac{\partial\phi(x,y)}{\partial x}$, $F_2 = \frac{\partial\phi(x,y)}{\partial y}$, $\frac{\partial F_1}{\partial y} = \frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_2}{\partial x}$

By Green's theorem, $\oint_C F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = 0$, ∴ $\vec{F}$ is conservative on Ω.

Eg. Is $\vec{F}(x,y,z) = -\frac{y}{z}\hat{i} - \frac{x}{z}\hat{j} + \frac{xy}{z^2}\hat{k}$ conservative in the region *z*>0? 【成大化工所】

(Sol.) $\nabla\times\vec{F} = 0$ at *z*>0, ∴ $\vec{F}$ is conservative.

Eg. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = -\frac{y}{z}\hat{i} - \frac{x}{z}\hat{j} + \frac{xy}{z^2}\hat{k}$, *C* is a piecewise smooth curve from (1,1,1) to (2,-1,3) and not crossing the *xy*-plane. 【成大化工所】

(Sol.) $\nabla\times\vec{F} = 0$, $\therefore \exists\phi(x,y,z)$ such that $\vec{F} = \nabla\phi$

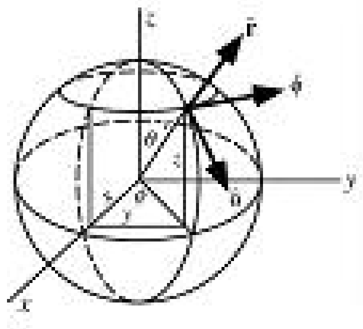
$$\Rightarrow \phi(x,y,z) = -\frac{xy}{z} \Rightarrow \int \vec{F}\cdot d\vec{r} = \phi(x,y,z)\Big|_{(1,1,1)}^{(2,-1,3)} = \frac{5}{3}$$

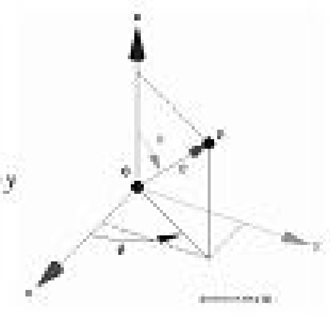
7-6 Curvilinear Coordinates

Coordinate transformation: $\begin{cases} x = x(q_1,q_2,q_3) \\ y = y(q_1,q_2,q_3) \\ z = z(q_1,q_2,q_3) \end{cases} \Leftrightarrow \begin{cases} q_1 = q_1(x,y,z) \\ q_2 = q_2(x,y,z) \\ q_3 = q_3(x,y,z) \end{cases}$

Scalar factors: If (q_1,q_2,q_3) are orthogonal system, then

$$h_{ij} = \begin{cases} 0, i \neq j \\ \left[\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2\right]^{1/2}, i = j \end{cases}$$



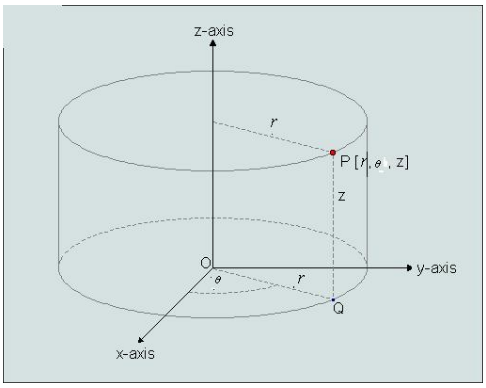


Eg. Rectangular coordinate $\Leftrightarrow$ Spherical coordinate

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \sin^{-1}\left(\sqrt{x^2+y^2}\Big/\sqrt{x^2+y^2+z^2}\right) \\ \phi = \cos^{-1}\left(x\Big/\sqrt{x^2+y^2}\right) \end{cases} \Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1 \\ h_\theta = h_{22} = \left[\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right]^{1/2} = r \\ h_\phi = h_{33} = \left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2\right]^{1/2} = r\sin\theta \end{cases}$$

Differential length vector in the spherical coordinate:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r\sin\theta d\phi$$



Eg. Rectangular coordinate $\Leftrightarrow$ Cylindrical coordinate

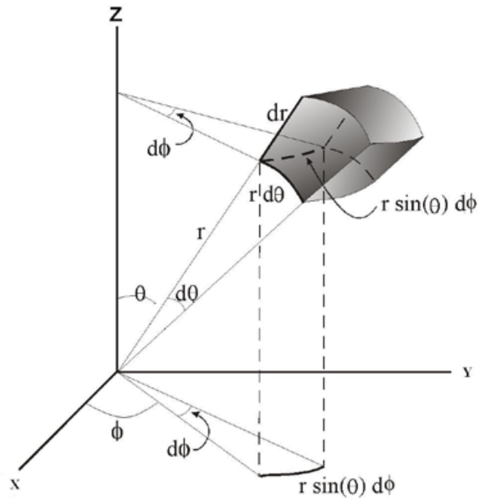
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases} \Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1 \\ h_\theta = h_{22} = \left[\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right]^{1/2} = r \\ h_z = h_{33} = \left[\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2\right]^{1/2} = 1 \end{cases}$$

Differential length vector in the cylindrical coordinate: $d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_z dz$

Differential arc length: $ds = [(h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2]^{1/2}$

Differential elements of area on the q_iq_j-plane: $dA_{ij} = ds_i\ ds_j = h_ih_jdq_idq_j$

Differential element of volume: $dV = ds_1ds_2ds_3 = h_1h_2h_3dq_1dq_2dq_3$



Eg. Spherical coordinate:

$$\begin{cases} ds = \left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]^{1/2} \\ dA_{\theta,\phi} = r^2\sin\theta d\theta d\phi, dA_{r,\theta} = rdrd\theta, etc. \\ dV = r^2\sin\theta drd\theta d\phi \end{cases}$$

Eg. Cylindrical coordinate:
$$\begin{cases} ds = \left[dr^2 + r^2d\theta^2 + dz^2\right]^{1/2} \\ dA_{\theta,z} = rd\theta dz, dA_{r,\theta} = rdrd\theta, dA_{r,z} = drdz \\ dV = rdrd\theta dz \end{cases}.$$

Jacobian determinants: $(p_1,p_2,p_3) \Leftrightarrow (q_1,q_2,q_3)$

$$dp_idp_j = \left|\det\begin{bmatrix} \partial p_i/\partial q_i & \partial p_j/\partial q_i \\ \partial p_i/\partial q_j & \partial p_j/\partial q_j \end{bmatrix}\right| \cdot dq_idq_j$$

and $$dp_1dp_2dp_3 = \left|\det\begin{bmatrix} \partial p_1/\partial q_1 & \partial p_2/\partial q_1 & \partial p_3/\partial q_1 \\ \partial p_1/\partial q_2 & \partial p_2/\partial q_2 & \partial p_3/\partial q_2 \\ \partial p_1/\partial q_3 & \partial p_2/\partial q_3 & \partial p_3/\partial q_2 \end{bmatrix}\right| \cdot dq_1dq_2dq_3$$

Eg. Let $I=\iiint_V f(x,y,z)dxdydz$. Transform the integral from (x,y,z) into (r,θ,φ).

$$|J| = \left|\det\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{bmatrix}\right| = \left| r^2\sin\theta\cos^2\theta\cos^2\phi \right.$$

$$\left. + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi + r^2\sin^3\theta\cos^2\phi \right| = \left| r^2\sin\theta\cos^2\theta + r^2\sin^3\theta \right| = r^2\sin\theta = h_1h_2h_3$$

$$\therefore\ I = \iiint_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2\sin\theta drd\theta d\phi$$

Del operators in (q_1,q_2,q_3) coordinate system:

$$\nabla\psi(q_1,q_2,q_3)=\frac{1}{h_1}\frac{\partial\psi}{\partial q_1}\hat{u}_1+\frac{1}{h_2}\frac{\partial\psi}{\partial q_2}\hat{u}_2+\frac{1}{h_3}\frac{\partial\psi}{\partial q_3}\hat{u}_3$$

$$\nabla\cdot\vec{F}(q_1,q_2,q_3)=\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}(F_1h_2h_3)+\frac{\partial}{\partial q_2}(F_2h_1h_3)+\frac{\partial}{\partial q_3}(F_3h_1h_2)\right)$$

$$\nabla\times\vec{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}$$

$$\nabla^2\psi(q_1,q_2,q_3)=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right],$$

where $\vec{F}=F_1\hat{u}_1+F_2\hat{u}_2+F_3\hat{u}_3$

Eg. Spherical coordinate system: $h_r=1$, $h_\theta=r$, $h_\varphi=r\sin\theta$

$$\vec{F}=F_r\hat{u}_r+F_\theta\hat{u}_\theta+F_\phi\hat{u}_\phi$$

$$\nabla\cdot\vec{F}=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(\sin\theta\cdot F_\theta)+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(F_\phi)$$

$$\nabla\times\vec{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{u}_r & r\hat{u}_\theta & r\sin\theta\hat{u}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta\cdot F_\phi \end{vmatrix},\quad \nabla\psi=\frac{\partial\psi}{\partial r}\hat{u}_r+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{u}_\theta+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{u}_\phi$$

$$\nabla^2\psi=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\phi^2}$$

Eg. Cylindrical coordinate system: $h_r=1$, $h_\theta=r$, $h_z=1$

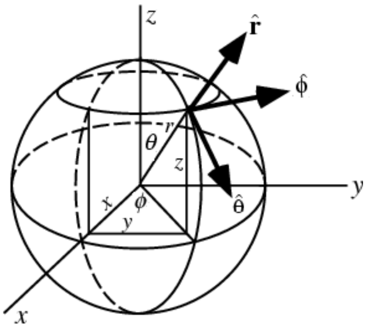
$$\vec{F}=F_r\hat{u}_r+F_\theta\hat{u}_\theta+F_z\hat{u}_z$$

$$\nabla\cdot\vec{F}=\frac{1}{r}\frac{\partial}{\partial r}(rF_r)+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial z}$$

$$\nabla\times\vec{F}=\left(\frac{1}{r}\frac{\partial F_z}{\partial\theta}-\frac{\partial F_\theta}{\partial z}\right)\hat{u}_r+\left(\frac{\partial F_\theta}{\partial z}-\frac{\partial F_z}{\partial r}\right)\hat{u}_\theta+\left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta)-\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right)\hat{u}_z$$

$$\nabla\psi=\frac{\partial\psi}{\partial r}\hat{u}_r+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{u}_\theta+\frac{\partial\psi}{\partial z}\hat{u}_z,\quad \nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}$$

Unit vector conversions between distinct coordinates:



Eg. Rectangular coordinate ⇔ Spherical coordinate

(Proof) $\hat{x}\cdot\hat{r}=\sin\theta\cos\phi$, $\hat{y}\cdot\hat{r}=\sin\theta\sin\phi$,

$\hat{z}\cdot\hat{r}=\cos\theta$, ∴ $\hat{r}=\hat{x}\sin\theta\cos\phi+\hat{y}\sin\theta\sin\phi+\hat{z}\cos\theta$

Similarly, $\hat{x}\cdot\hat{\theta}=\cos\theta\cos\phi$, $\hat{y}\cdot\hat{\theta}=\cos\theta\sin\phi$,

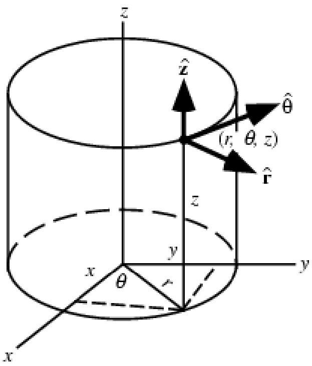
$\hat{z}\cdot\hat{\theta}=-\sin\theta$, ∴ $\hat{\theta}=\hat{x}\cos\theta\cos\phi+\hat{y}\cos\theta\sin\phi-\hat{z}\sin\theta$,

And $\hat{x}\cdot\hat{\phi}=-\sin\phi$, $\hat{y}\cdot\hat{\phi}=\cos\phi$, $\hat{z}\cdot\hat{\phi}=0$,

∴ $\hat{\phi}=-\hat{x}\sin\phi+\hat{y}\cos\phi$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\cdot\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix},$$

or $$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\cdot\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}$$



Eg. Rectangular coordinate ⇔ Cylindrical coordinate

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{z}\end{bmatrix}$$ or

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{z}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$$