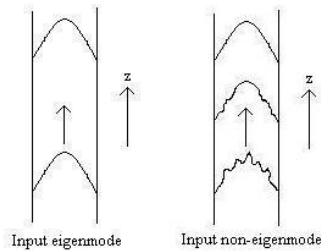
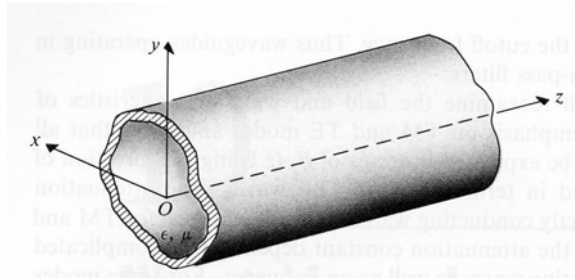


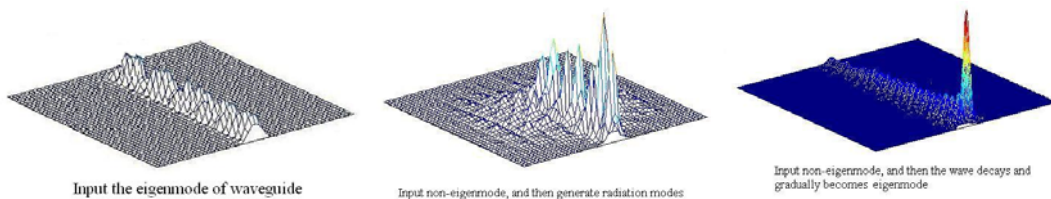
# Chapter 6 Metallic Waveguide and Cavity Resonators

## 6-1 General Metallic Waveguides



How to study the theory of metallic waveguides (by L. J. Chu, 朱蘭成):

1. Specify a proper coordinate system, and derive **waveguide's equations** to express the transverse components of the  $E$ - and  $H$ -fields in terms of the longitudinal components by Maxwell's equations.
2. Calculate the eigenmodes (**TM mode, TE mode, TEM mode or other types of modes**) of the waveguide, and obtain the eigenvalues and the longitudinal field-components of the corresponding eigenmodes by solving the wave equations. Substituting the longitudinal field-components into the longitudinal components, we can obtain the other components. If the eigenmode is injected into a waveguide, it can propagate along an infinitely-long straight waveguide without any deformation. However, in case the input EM wave is not an eigenmode, some power loss occurs and then it becomes the eigenmode gradually. All the eigenmodal functions in an infinitely-long straight metallic waveguide are orthogonal to each other. Moreover, these eigenmodes form a complete set (a basis in a vector space), such that any electromagnetic fields within the waveguide can be uniquely expressed by the eigenmodal functions.
3. Obtain the quantities of the physical characteristics for a given eigenmode, such as the **cutoff frequency ( $f_c$ )**, the **propagation constant ( $\gamma = \alpha + j\beta$ )**, the **phase velocity ( $v_p = \omega/\beta$ )**, the **group velocity ( $v_g = \partial\omega/\partial\beta$ )**, the impedance  $Z$ , etc.



From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ :	From $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$ :
$\frac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$	$\frac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\epsilon E_x^0$
$-\gamma E_x^0 - \frac{\partial E_z^0}{\partial x} = -j\omega\mu H_y^0$	$-\gamma H_x^0 - \frac{\partial H_z^0}{\partial x} = j\omega\epsilon E_y^0$
$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0$	$\frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = j\omega\epsilon E_z^0$

**Waveguide's equations:** According to Ampere's law and Faraday's law, we obtain

$$H_x^0 = -\frac{1}{h^2}(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y}), \quad H_y^0 = -\frac{1}{h^2}(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x})$$

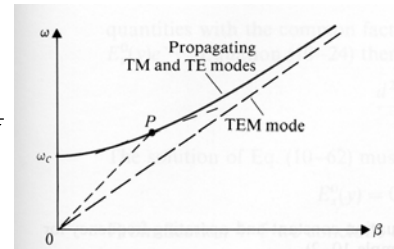
$$E_x^0 = -\frac{1}{h^2}(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}), \quad E_y^0 = -\frac{1}{h^2}(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}), \quad \text{where } h^2 = \gamma^2 + k^2,$$

$$\text{and } \begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \end{cases} \Rightarrow \begin{cases} \nabla_t^2 \vec{E} + (\gamma^2 + k^2) \vec{E} = 0 = [\nabla_t^2 + h^2] \vec{E} \\ \nabla_t^2 \vec{H} + (\gamma^2 + k^2) \vec{H} = 0 = [\nabla_t^2 + h^2] \vec{H} \end{cases}$$

**Case 1 TEM mode:**  $E_z = H_z = 0$

$$h^2 = 0 = \gamma_{TEM}^2 + k^2 \Rightarrow \gamma_{TEM} = jk = j\omega\sqrt{\mu\epsilon}, \quad v_p = \frac{1}{\sqrt{\mu\epsilon}}$$

$$Z_{TEM} = \frac{E_x^0}{H_y^0} = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{\gamma_{TEM}}{j\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta \quad \text{and} \quad \vec{H} = \frac{1}{Z_{TEM}} \hat{z} \times \vec{E}$$



**Note:** All frequencies make  $\gamma_{TEM}$  is pure imaginary  $\Rightarrow$  TEM wave can propagate at any frequency, no cutoff

**Case 2 TM mode:**  $H_z = 0, E_z \neq 0$  and  $\nabla_t^2 E_z + h^2 E_z = 0$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial y}, \quad H_y^0 = -\frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial x}, \quad E_x^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial x}, \quad E_y^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial y}$$

$$\Rightarrow Z_{TM} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{\gamma}{j\omega\epsilon} (\neq \frac{j\omega\mu}{\gamma}) \quad \text{and} \quad \vec{H} = \frac{1}{Z_{TM}} (\hat{z} \times \vec{E})$$

$$\gamma = \sqrt{h^2 - k^2} = h \sqrt{1 - \left(\frac{f}{f_c}\right)^2}, \quad \text{where } f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}}$$

$$\text{If } f > f_c, \quad \gamma = j\beta = j\omega\sqrt{\mu\epsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \Rightarrow Z_{TM} = \eta \sqrt{1 - \left(\frac{f_c}{f}\right)^2}, \quad v_p = \frac{1}{\sqrt{\mu\epsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

**Case 3 TE mode:**  $E_z=0, H_z \neq 0$  and  $\nabla_{\perp}^2 H_z + h^2 H_z = 0$

$$H_x^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial x}, \quad H_y^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial y}, \quad E_x^0 = -\frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial y}, \quad E_y^0 = \frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial x}$$

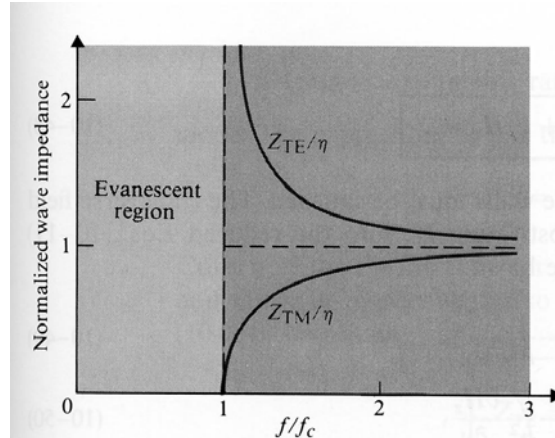
$$\Rightarrow Z_{TE} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma} \left( \neq \frac{\gamma}{j\omega\epsilon} \right) \quad \text{and} \quad \vec{E} = -Z_{TE} (\hat{z} \times \vec{H})$$

If  $f > f_c$ ,  $\gamma = j\beta = jk \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = j\omega \sqrt{\mu\epsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

$$\Rightarrow Z_{TE} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}}, \quad v_p = \frac{1}{\sqrt{\mu\epsilon} \sqrt{1 - (f_c/f)^2}}$$

**Wave Impedances and Guide Wavelengths for  $f > f_c$**

Mode	Wave Impedance, Z	Guide Wavelength, $\lambda_g$
TEM	$\eta = \sqrt{\frac{\mu}{\epsilon}}$	$\lambda = \frac{1}{f\sqrt{\mu\epsilon}}$
TM	$\eta \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$	$\frac{\lambda}{\sqrt{1 - (f_c/f)^2}}$
TE	$\frac{\eta}{\sqrt{1 - (f_c/f)^2}}$	$\frac{\lambda}{\sqrt{1 - (f_c/f)^2}}$



A case of longitudinal  $v_p > 0$  but longitudinal  $v_g = 0$  in barber's pole.

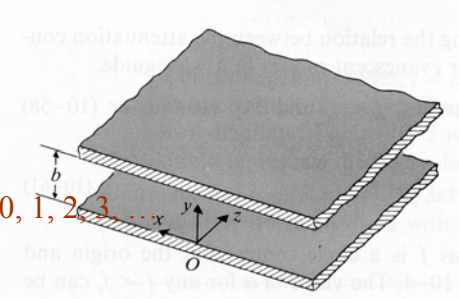


## 6-2 Parallel-Plate Waveguides

**Case 1  $TM_n$  mode:**  $H_z=0$ ,  $E_z(x,y) = E_z^0(y)e^{-\gamma z}$

$$\frac{dE_z^0(y)}{dy^2} + h^2 E_z^0(y) = 0, \quad E_z^0(y) = 0 \quad \text{at } y=0 \text{ and } b$$

$$\Rightarrow \text{Eigenvalues: } h = \frac{n\pi}{b}, \quad E_z^0(y) = A_n \sin\left(\frac{n\pi y}{b}\right), \quad n=0, 1, 2, 3, \dots$$



$$\begin{cases} H_x^0(y) = \frac{j\omega\epsilon}{h^2} \frac{dE_z^0(y)}{dy} = \frac{j\omega\epsilon}{h} A_n \cos\left(\frac{n\pi y}{b}\right) \\ E_y^0(y) = -\frac{\gamma}{h^2} \frac{dE_z^0(y)}{dy} = -\frac{h}{\gamma} A_n \cos\left(\frac{n\pi y}{b}\right) \end{cases} \quad \text{and } \gamma = \sqrt{h^2 - \omega^2 \mu\epsilon} = \sqrt{\left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu\epsilon}$$

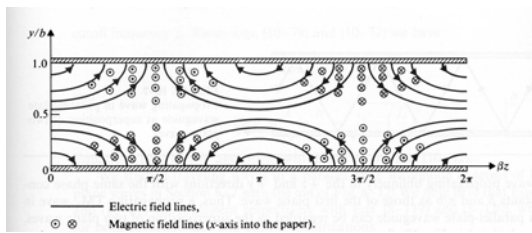
Cutoff frequency:  $f_c = \frac{n}{2b\sqrt{\mu\epsilon}}$  fulfills  $\gamma=0$ . (Note:  $n=0$  is the TEM mode)

**Case 2  $TE_n$  modes:**  $E_z=0$ ,  $H_z(y,z) = H_z^0(y)e^{-\gamma z}$

$$\frac{d^2 H_z^0(y)}{dy^2} + h^2 H_z^0(y) = 0, \quad \frac{dH_z^0(y)}{dy} = 0 \quad \text{at } y=0 \text{ and } b \Rightarrow H_z^0(y) = B_n \cos\left(\frac{n\pi y}{b}\right) \quad \text{and}$$

$$f_c = \frac{n}{2b\sqrt{\mu\epsilon}}$$

$$\begin{cases} H_y^0(y) = -\frac{j\omega\epsilon}{h^2} \frac{dH_z^0(y)}{dy} = \frac{\gamma}{h} B_n \sin\left(\frac{n\pi y}{b}\right) \\ E_x^0(y) = -\frac{\gamma}{h^2} \frac{dH_z^0(y)}{dy} = \frac{j\omega\mu}{h} B_n \sin\left(\frac{n\pi y}{b}\right) \end{cases} \quad \text{and } \gamma = \sqrt{\left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu\epsilon}, \quad n=1, 2, 3, \dots$$



**Fig. (a)** Write the instantaneous field expression for  $TM_1$  mode in a parallel-plate waveguide. **(b)** Sketch the  $E$ - &  $H$ - field lines in the  $yz$ -plane.

(Sol.) (a) For  $n=1$ ,  $E_y(y,z,t) = \frac{\beta b}{\pi} A_1 \cos\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$

$$H_z(y,z,t) = -\frac{\omega\epsilon b}{\pi} A_1 \cos\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z), \quad \beta = \sqrt{\omega^2 \mu\epsilon - \left(\frac{\pi}{b}\right)^2}$$

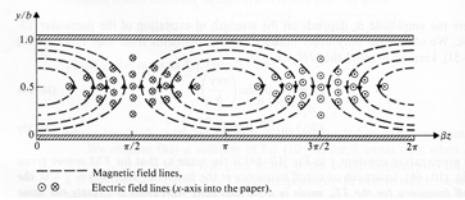
(b)  $\frac{dy}{E_y} = \frac{dz}{E_z} \Rightarrow \frac{dy}{dz} = \frac{E_y(y,z,0)}{E_z(y,z,0)} = -\frac{\beta b}{\pi} \cot\left(\frac{\pi y}{b}\right) \tan \beta z \Rightarrow \cos\left(\frac{\pi y}{b}\right) \cos \beta z = \text{constant}$

**Eg. (a) Write the instantaneous field expression for  $TE_1$  mode in a parallel-plate waveguide. (b) Sketch the electric and magnetic field lines in the  $yz$ -plane.**

(Sol.) (a) For  $n=1$ ,  $H_z(y, z; t) = B_1 \cos\left(\frac{\pi y}{b}\right) \cos(\omega t - \beta z)$ ,

$$H_y(y, z; t) = -\frac{\beta b}{\pi} B_1 \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$$

$E_x(y, z; t) = -\frac{\omega \mu b}{\pi} B_1 \sin\left(\frac{\pi y}{b}\right) \sin(\omega t - \beta z)$ , where  $\beta$  is the same as that of the  $TM_1$  mode.



(b) At  $t=0$ ,

$$E_x(y, z; 0) = -\frac{\omega \mu b}{\pi} B_1 \sin\left(\frac{\pi y}{b}\right) \sin \beta z,$$

$$\frac{dy}{dz} = \frac{H_y(y, z; 0)}{H_z(y, z; 0)} = \frac{\beta b}{\pi} \tan\left(\frac{\pi y}{b}\right) \tan \beta z$$

**Eg. Find the electric and the magnetic fields of the propagating wave in a parallel waveguide  $b=5\text{cm}$ , filled with a dielectric ( $4\epsilon_0, \mu_0$ ) and excited by  $\vec{H} = \hat{y} \cos 40\pi x \sin 8\pi 10^9 t$ .**

(Sol.)  $f=4 \times 10^9 \text{ Hz}$ ,  $\cos 40\pi x = \cos\left(\frac{2\pi x}{0.05}\right)$ :  $TM_2$  mode

$$(f_c)_{TM_2} = \frac{2}{2b\sqrt{\mu_0 4\epsilon_0}} = 3 \times 10^9, (f_c)_{TM_2} < f, \therefore TM_2 \text{ mode can propagate!}$$

$$\beta = \omega \sqrt{\mu_0 4\epsilon_0} \sqrt{1 - \left(\frac{(f_c)_{TM_2}}{f}\right)^2} = 110.82 \Rightarrow \vec{H} = \hat{y} \cos(40\pi x) \sin(8\pi 10^9 t - 110.82 z)$$

$$\vec{E} = \hat{x} 124.67 \cos(40\pi x) \sin(8\pi 10^9 t - 110.82 z) + \hat{z} 141.37 \sin(40\pi x) \cos(8\pi 10^9 t - 110.82 z)$$

**Eg. Find the electric field of the propagating wave in an air parallel waveguide  $b=5\text{cm}$  excited by  $\vec{E} = \hat{y} 10(\sin 20\pi x + 0.5 \sin 60\pi x) \sin 10^{10} \pi t$ .**

(Sol.)  $f=5 \times 10^9 \text{ Hz}$ ,  $\sin 20\pi x = \sin\left(\frac{\pi x}{0.05}\right)$ :  $TE_1$  mode,  $\sin 60\pi x = \sin\left(\frac{3\pi x}{0.05}\right)$ :  $TE_3$  mode

$$(f_c)_{TE_1} = \frac{1}{2b\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^9, (f_c)_{TE_3} = \frac{3}{2b\sqrt{\mu_0 \epsilon_0}} = 9 \times 10^9, (f_c)_{TE_1} < f < (f_c)_{TE_3}$$

$$\therefore \text{Only } TE_1 \text{ mode can propagate! } \beta = \omega \sqrt{\mu_0 \epsilon_0} \sqrt{1 - \left(\frac{(f_c)_{TE_1}}{f}\right)^2} = \frac{80\pi}{3}$$

$$\vec{E} = \hat{y} 10 \sin(20\pi x) \sin\left(10^{10} \pi t - \frac{80\pi z}{3}\right)$$

## Energy-transport velocity and attenuation in parallel-plate waveguides:

**Energy velocity:**  $v_{en} = \frac{(P_z)}{W'_{av}}$  and  $(v_{en})_{TE} = (v_{en})_{TM}$

### Energy velocity of TM mode:

$$(P_z)_{av} = \int_s \bar{P}_{av} \cdot d\vec{S} = \int_s \frac{1}{2} \text{Re}[\bar{E} \times \bar{H}^*] \cdot \hat{z} dy = \int_s \frac{1}{2} \text{Re}(-\hat{z} E_y^0 H_x^0 + \hat{y} E_z^0 H_x^0) \cdot \hat{z} dy$$

$$= \frac{\omega \varepsilon \beta b}{4h^2} A_n^2$$

$$w'_{av} = \frac{\varepsilon}{4} \text{Re}(\bar{E} \cdot \bar{E}^*) + \frac{\mu}{4} \text{Re}(\bar{H} \cdot \bar{H}^*) = \frac{\varepsilon A_n^2}{4} \left[ \sin^2\left(\frac{n\pi y}{b}\right) + \frac{\beta^2}{h^2} \cos^2\left(\frac{n\pi y}{b}\right) \right] + \frac{\mu}{4} \left(\frac{\omega^2 \varepsilon^2}{h^2}\right) A_n^2 \cos\left(\frac{n\pi y}{b}\right)$$

$$W'_{av} = \int_0^b w'_{av} dy = \frac{\varepsilon b}{4h^2} k^2 A_n^2 \Rightarrow v_{en} = \frac{\omega \beta}{k^2} = \frac{1}{\sqrt{\mu \varepsilon}} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

**Attenuation constant:**  $\alpha = \alpha_d + \alpha_c$

**$\alpha$  of TEM mode:**  $\alpha_d = \frac{G}{2} R_0 = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2} \eta \approx \frac{\omega \varepsilon''}{2} \eta$ ,  $\alpha_c = \frac{R}{2R_0} = \frac{1}{b} \sqrt{\frac{\pi f \varepsilon}{\sigma_c}}$

**$\alpha$  of TM mode:**

$$\gamma = j \left[ \omega^2 \mu \varepsilon \left(1 - \frac{j\sigma}{\omega \varepsilon}\right) - \left(\frac{n\pi}{b}\right)^2 \right]^{1/2} \approx j \sqrt{\omega^2 \mu \varepsilon - \left(\frac{n\pi}{b}\right)^2} \cdot \left\{ 1 - \frac{j\omega \mu \sigma}{2} \left[ \omega^2 \mu \varepsilon - \left(\frac{n\pi}{b}\right)^2 \right]^{-1} \right\}$$

$$= j \omega \sqrt{\mu \varepsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \cdot \left\{ 1 - \frac{j\omega \mu \sigma}{2} \cdot \frac{1}{1 + \left(\frac{f_c}{f}\right)^2} \right\} = \alpha_d + j\beta \Rightarrow \alpha_d = \frac{\sigma \eta}{2 \sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\beta = \omega \sqrt{\mu \varepsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\alpha_c = \frac{P_L(z)}{2P(z)}$$
, where  $P(z) = w \int_0^b -\frac{1}{2} (E_y^0)(H_x^0)^* dy = w \omega \varepsilon \beta b \left(\frac{b A_n}{2n\pi}\right)^2$ ,

$$|J_{sz}^0| = |H_x^0(y=0)| = \frac{\omega \varepsilon b A_n}{n\pi} \Rightarrow P_L(z) = 2w \left(\frac{1}{2} |J_{sz}^0|^2 R_s\right) = w \left(\frac{\omega \varepsilon b A_n}{n\pi}\right)^2 R_s$$
,

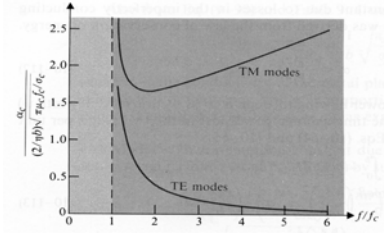
$$R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} \Rightarrow \alpha_c = \frac{2}{\eta b} \cdot \sqrt{\frac{\pi \mu_c f_c}{\sigma_c}} \cdot \frac{1}{\sqrt{(f_c/f)[1 - (f_c/f)^2]}}$$

**$\alpha$  of TE mode:**  $\alpha_d$  is the same as the expression in TM mode

$$P(z) = w \int_0^b \frac{1}{2} (E_x^0)(H_y^0) * dy = \frac{w\omega\mu\beta}{2} \left(\frac{bB_n}{n\pi}\right)^2 \int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy = w\omega\mu\beta b \left(\frac{bB_n}{2n\pi}\right)^2,$$

$$\beta = \omega\sqrt{\mu\epsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$P_L(z) = 2w \left(\frac{1}{2} |J_{sx}^0|^2 R_s\right) = w |H_z^0(y=0)|^2 R_s = wB^2 R_s$$



$$\alpha_c = \frac{P_L(z)}{2P(z)} = \frac{2R_s}{\omega\mu\beta b} \left(\frac{n\pi}{b}\right)^2 = \frac{2R_s f_c^2}{\eta b f^2 \sqrt{1 - (f_c/f)^2}} = \frac{2}{\eta b} \sqrt{\frac{\pi\mu_c f_c}{\sigma_c}} \sqrt{\frac{(f_c/f)}{1 - (f_c/f)^2}}$$

**Note:**  $\therefore \left\{ \begin{matrix} \alpha_c \\ \alpha_d \end{matrix} \right\}$  of the higher-order modes  $> \left\{ \begin{matrix} \alpha_c \\ \alpha_d \end{matrix} \right\}$  of the lower-order modes,  $\therefore$

the lowest-order mode is often utilized in communication systems. Otherwise, the signal decays very soon.

**Ex. A waveguide is formed by two parallel copper sheets, which is separated by a 5cm thick lossy dielectric  $\epsilon_r=2.25$ ,  $\mu_r=1$ ,  $\sigma=10^{-10}$  (S/m). For an operating frequency of 10GHz, find  $\alpha_d$ ,  $\alpha_c$ ,  $\beta$ ,  $v_p$ ,  $v_g$  and  $\lambda_g$  for (a) the TEM mode, (b) the  $TM_1$  mode.**

(Sol.)  $\sigma_c=5.8 \times 10^7$  (S/m)

(a) TEM:  $\beta = \omega\sqrt{\mu\epsilon} = 314.16 \text{ rad/m}$ ,  $\alpha_d = \frac{\sigma\eta}{2} = 1.257 \times 10^{-8} \text{ Np/m}$ ,

$$\alpha_c = \frac{1}{b} \sqrt{\frac{\pi f \epsilon}{\sigma_c}} = 2.078 \times 10^{-3} \text{ Np/m}, \quad v_p = \frac{1}{\sqrt{\mu\epsilon}} = 2 \times 10^8 \text{ m/s}, \quad v_g = v_p,$$

$$\lambda_g = \frac{v_p}{f} = 0.02 \text{ m}$$

(b)  $TM_1$ :  $f_c = \frac{1}{2b\sqrt{\mu\epsilon}} = 2 \times 10^9 \text{ Hz}$ ,  $\beta = \omega\sqrt{\mu\epsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 307.88 \text{ rad/m}$

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}} = 1.282 \times 10^{-8} \text{ Np/m}, \quad \alpha_c = \frac{1}{\eta b} \cdot \frac{1}{\sqrt{(f_c/f)[1 - (f_c/f)^2]}} = 4.237 \times 10^{-3} \text{ Np/m}$$

$$v_p = \frac{\omega}{\beta} = 2.041 \times 10^8 \text{ m/s}, \quad v_g = \frac{(3 \times 10^8)^2}{2.25 v_p} = 1.96 \times 10^8 \text{ m/s}, \quad \lambda_g = \frac{v_p}{f} = 0.0204 \text{ m}$$

**Eg. A parallel-plate waveguide made of two perfectly conducting infinite planes spaced 3cm apart in air operates at a frequency 10GHz. Find the maximum time-average power that can be propagated per unit width of the without a voltage breakdown for (a) the TEM mode, (b) the  $TM_1$  mode, (c) the  $TE_1$  mode.**

(Sol.) Without breakdown:  $E_{\max}=3\times 10^6 V/m$ ,  $b=3\times 10^{-2}m$ ,  $f_c = \frac{1}{2b\sqrt{\mu\epsilon}} = 5\times 10^9 (Hz)$ ,

$$\beta = \omega\sqrt{\mu\epsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\text{TEM: } P = \frac{1}{2} \text{Re}(E_y H_x^*) \cdot b = \frac{bE_{\max}^2}{2\eta} = 3.581 \times 10^8$$

$$TM_1: \frac{\beta b}{\pi} A_1 = 3 \times 10^6 \Rightarrow A_1 = \frac{3 \times 10^6 \pi}{\beta b}, \quad P_{av} = \omega\epsilon\beta b \left(\frac{bA_1}{2\pi}\right)^2 = 2.067 \times 10^8 W$$

$$TE_1: \frac{\omega\mu b}{\pi} B_1 = 3 \times 10^6 \Rightarrow B_1 = \frac{3 \times 10^6 \pi}{\omega\mu b} \Rightarrow P_{av} = \omega\mu\beta b \left(\frac{bB_1}{2\pi}\right) = 1.55 \times 10^8 W$$





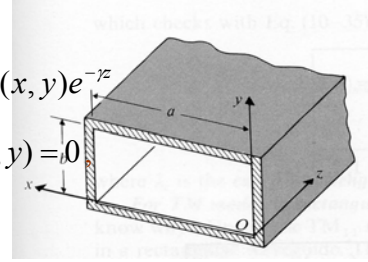
## 6-3 Rectangular Waveguides

**Case 1  $TM_{mn}$  modes:**  $H_z=0$ ,  $E_z(x, y, z) = E_z^0(x, y)e^{-\gamma z}$

$$(\nabla_t^2 + h^2)E_z^0(x, y) = 0 \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2\right)E_z^0(x, y) = 0$$

$$E_z^0(x, y) = 0 \text{ at } x=0, a, \text{ and } y=0, b$$

**Eigenvalues:**  $h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ ,  $E_z^0(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$



Waveguide's equations  $\Rightarrow$

$$\begin{cases} E_x^0(x, y) = -\frac{\gamma}{h^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), & E_y^0(x, y) = -\frac{\gamma}{h^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \\ H_x^0(x, y) = \frac{j\omega\epsilon}{h^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), & H_y^0(x, y) = -\frac{j\omega\epsilon}{h^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \end{cases}$$

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \text{ Note: } TM_{mn} \text{ mode, neither } m \text{ nor } n \text{ can be zero.}$$

**Cutoff frequency:**  $(f_c)_{TM_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

In case of  $f > f_c$ : waves can propagate, else if  $f < f_c$ : evanescent waves (cutoff).

**Case 2  $TE_{mn}$  modes:**  $E_z=0$ ,  $H_z(x, y, z) = H_z^0(x, y)e^{-\gamma z}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2\right)H_z^0(x, y) = 0, \begin{cases} \frac{\partial H_z^0}{\partial x} = 0 \text{ (or } E_y = 0) & x = 0, a \\ \frac{\partial H_z^0}{\partial y} = 0 \text{ (or } E_x = 0) & y = 0, b \end{cases} \text{ at}$$

$$\Rightarrow H_z^0(x, y) = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

Waveguide's equations  $\Rightarrow$

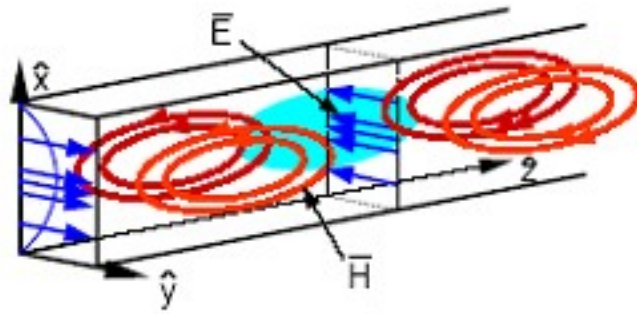
$$\begin{cases} E_x^0(x, y) = \frac{j\omega\mu}{h^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), & E_y^0(x, y) = -\frac{j\omega\mu}{h^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \\ H_x^0(x, y) = \frac{\gamma}{h^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), & H_y^0(x, y) = \frac{\gamma}{h^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \end{cases}$$

where  $\gamma = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$  and  $(f_c)_{TE_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

In case of  $2b > a > b$ , the fundamental mode of the rectangular waveguide is  $TE_{10}$  mode.

It has the lowest cutoff frequency  $(f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{v}{2a}$  (Hz)

TE<sub>10</sub> waveguide mode

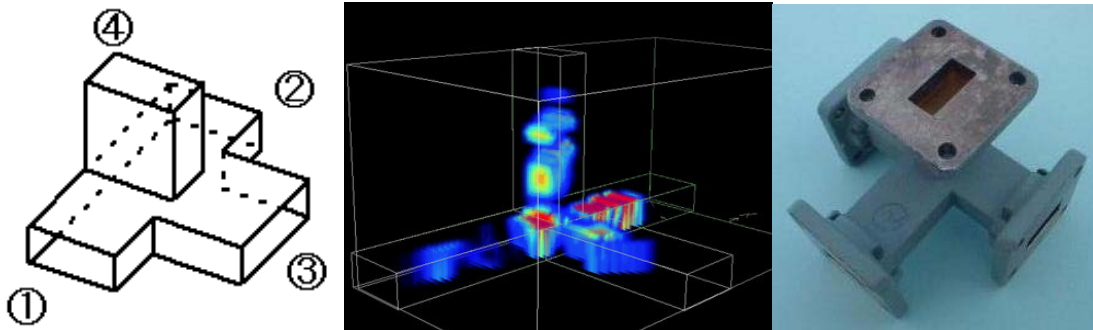


**Note:** We prefer a waveguide that is **single-mode** because the **fundamental mode** has the **lowest attenuation** and **avoid modal dispersion**. The **bandwidth** of the single-mode waveguide is  $(f_c)_1 < f < (f_c)_2$ .

**Dispersion:** Waves have different velocities

**Modal Dispersion:** Different modes have distinct phase velocities.

**Magic-T-junction:** (made of rectangular metallic waveguide)



**Eg. Calculate and compare the values of  $\beta$ ,  $v_p$ ,  $v_g$ ,  $\lambda_g$  and  $Z_{TE_{10}}$  for a  $2.5\text{cm} \times 1.5\text{cm}$  rectangular waveguide operating at  $7.5\text{GHz}$ . if the waveguide is hollow.**

$$\text{(Sol.) } \mu = \mu_0, \varepsilon = \varepsilon_0, (f_c)_{TE_{10}} = \frac{1}{2a} \frac{1}{\sqrt{\mu\varepsilon}} = 6 \times 10^9 \text{ Hz}, \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 0.6$$

$$\beta = \omega \sqrt{\mu\varepsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 94.25 \text{ rad/m}, v_p = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \frac{1}{\sqrt{1 - (f_c/f)^2}} = 5 \times 10^8 \text{ m/s}$$

$$\lambda_g = v_p / f = 0.067 \text{ m}, Z_{TE_{10}} = \frac{120\pi}{\sqrt{1 - (f_c/f)^2}} = 628.3 \Omega, v_g = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 1.8 \times 10^8 \text{ m/s}$$

**Eg. Calculate and list in ascending order the cutoff frequencies of an  $a \times b$  rectangular waveguide for the following modes:  $TE_{01}$ ,  $TE_{10}$ ,  $TE_{11}$ , and  $TE_{02}$  if  $a=2b$ .**

$$\text{(Sol.) } f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, a=2b \Rightarrow (f_c)_{TE_{10}} = \frac{1}{4b\sqrt{\mu\epsilon}}, (f_c)_{TE_{01}} = \frac{1}{2b\sqrt{\mu\epsilon}}$$

$$(f_c)_{TE_{11}} = (f_c)_{TM_{11}} = \frac{\sqrt{5}}{4b\sqrt{\mu\epsilon}}, (f_c)_{TE_{02}} = \frac{1}{b\sqrt{\mu\epsilon}}$$

**Eg. An air-filled  $a \times b$  ( $b < a < 2b$ ) rectangular waveguide is to be constructed to operate at 3GHz in the dominant mode. We desire the operating frequency to be at least 20% higher than the cutoff frequency of the dominant mode and also at least 20% below the cutoff frequency of the next higher-order mode. (a) Give a typical design for the dimensions  $a$  and  $b$ . (b) Calculate for your design  $\beta$ ,  $v_p$ ,  $\lambda_g$  and the wave impedance at the operating frequency. [台大電研]**

**(Sol.) (a)**  $f_c = \frac{1}{2\sqrt{\mu\epsilon}} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ .  $b < a < 2b$ , the dominant mode:  $TE_{10}$ , the next mode:  $TE_{01}$

$$(f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}}, (f_c)_{TE_{01}} = \frac{1}{2b\sqrt{\mu\epsilon}}, \frac{3 \times 10^9 - (1/2a\sqrt{\mu\epsilon})}{(1/2a\sqrt{\mu\epsilon})} > 20\%,$$

$$\frac{(1/2b\sqrt{\mu\epsilon}) - 3 \times 10^9}{(1/2b\sqrt{\mu\epsilon})} > 20\% \Rightarrow a \geq 0.06m, b \leq 0.04m, \text{ and } a < 2b$$

**(b)** Choose  $a=0.065m$ ,  $b=0.035m$ ,  $(f_c)_{TE_{10}} = 2.3 \times 10^9$  (Hz),  $\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 0.679$ ,

$$\beta = \omega\sqrt{\mu\epsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 40.15 \text{ rad/m}, v_p = \frac{1}{\sqrt{\mu\epsilon}} \cdot \frac{1}{\sqrt{1 - (f_c/f)^2}} = 4.7 \times 10^8 \text{ m/s},$$

$$\lambda_g = \frac{v_p}{f} = 0.157m, Z_{TE_{10}} = \eta_0 / \sqrt{1 - (f_c/f)^2} = 120\pi / 0.639 = 590\Omega$$

**Fig. A 3cm×1.5cm rectangular waveguide operating at 6GHz has a dielectric discontinuity between medium 1 ( $\mu_0, \epsilon_0$ ) and medium 2 ( $\mu_0, 4\epsilon_0$ ). (a) Find the SWR in the free-space region. (b) Find the length and the permittivity of a quarter-wave section to achieve a match between two media.**

(Sol.) For TE<sub>10</sub> mode,  $f_{c1} = \frac{1}{2a} \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 5 \times 10^9 \text{ Hz}$ ,  $f_{c2} = \frac{1}{2a} \frac{1}{\sqrt{\mu_0 4\epsilon_0}} = 2.5 \times 10^9 \text{ Hz}$

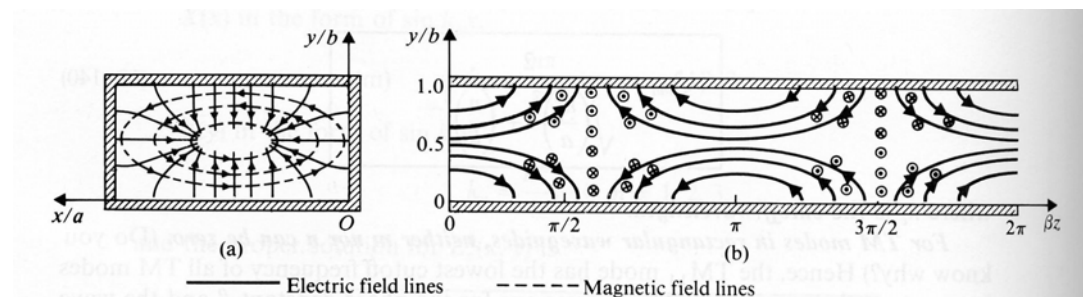
$$Z_1 = \sqrt{\mu_0 / \epsilon_0} / \sqrt{1 - (f_{c1}/f)^2} = 682 \Omega, \quad Z_2 = \sqrt{\mu_0 / 4\epsilon_0} / \sqrt{1 - (f_{c2}/f)^2} = 207 \Omega$$

$$\Rightarrow \Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -0.5337 \Rightarrow SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 3.289$$

$$f_{c3} = \frac{1}{2a} \frac{1}{\sqrt{\mu_0 \epsilon_r \epsilon_0}} \quad \text{and} \quad Z_3 = \sqrt{\mu_0 / \epsilon_r \epsilon_0} / \sqrt{1 - (f_{c3}/f)^2} = \sqrt{Z_1 Z_2} \Rightarrow \epsilon_r = 1.6995$$

$$d = \lambda_3 / 4 = v_{p3} / 4f = 1.24685 \times 10^{-2} \text{ m}$$

**Fig. (a) Write the instantaneous field expressions for the TM<sub>11</sub> mode in a rectangular waveguide of sides  $a$  and  $b$ . (b) Sketch the electric and magnetic field lines in a typical  $xy$ -plane and in a typical  $yz$ -plane.**



(Sol.) (a)  $m=1, n=1$

$$E_x(x, y, z; t) = \frac{\beta}{h^2} \left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z), \quad E_y(x, y, z; t) = \frac{\beta}{h^2} \left(\frac{\pi}{b}\right) E_0 \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z),$$

$$E_z(x, y, z; t) = E_0 \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} y\right) \cos(\omega t - \beta z), \quad H_x(x, y, z; t) = -\frac{\omega \epsilon}{h^2} \left(\frac{\pi}{b}\right) E_0 \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z),$$

$$H_y(x, y, z; t) = \frac{\omega \epsilon}{h^2} \left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z), \quad \text{where} \quad \beta = \sqrt{k^2 - h^2} = \sqrt{\omega^2 \mu \epsilon - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2}$$

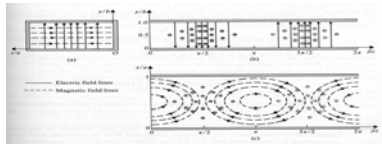
(b) In a typical  $xy$ -plane,  $\left(\frac{dy}{dx}\right)_E = \frac{a}{b} \tan\left(\frac{\pi}{a} x\right) \cot\left(\frac{\pi}{b} y\right)$ ,  $\left(\frac{dy}{dx}\right)_H = -\frac{b}{a} \cot\left(\frac{\pi}{a} x\right) \tan\left(\frac{\pi}{b} y\right)$

**Eg. For an  $a \times b$  rectangular waveguide operating at the  $TM_{11}$  mode, derive the expressions for the surface current densities on the conducting walls.**

$$(\text{Sol.}) \begin{cases} \vec{J}_s(x=0) = \hat{x} \times [\hat{y}(-\frac{j\omega\epsilon}{h^2}(\frac{\pi}{a})E_0 \sin(\frac{\pi y}{b}))] = -\hat{z} \frac{j\omega\epsilon}{h^2}(\frac{\pi}{a})E_0 \sin(\frac{\pi y}{b}), & \vec{J}_s(x=b) = \vec{J}_s(x=0) \\ \vec{J}_s(y=0) = \hat{y} \times [\hat{x} \frac{j\omega\epsilon}{h^2}(\frac{\pi}{b})E_0 \sin(\frac{\pi x}{a})] = -\hat{z} \frac{j\omega\epsilon}{h^2}(\frac{\pi}{b})E_0 \sin(\frac{\pi x}{a}), & \vec{J}_s(y=b) = \vec{J}_s(y=0) \end{cases}$$

**Eg. (a) Write the instantaneous field expression for the  $TE_{10}$  mode in a rectangular waveguide having sides  $a$  and  $b$ . (b) Sketch the electric and magnetic field lines in typical  $xy$ -,  $yz$ -, and  $xz$ -planes. (c) Sketch the surface currents on the guide walls.**

(Sol.) (a)  $m=1, n=0$ ,



$$E_x(x, y, z; t) = 0, \quad E_z(x, y, z; t) = 0, \\ E_y(x, y, z; t) = \frac{\omega\mu}{h^2}(\frac{\pi}{a})H_0 \sin(\frac{\pi}{a}x) \sin(\omega t - \beta z),$$

$$H_x(x, y, z; t) = -\frac{\beta}{h^2}(\frac{\pi}{a})H_0 \sin(\frac{\pi}{a}x) \cos(\omega t - \beta z), \quad H_y(x, y, z; t) = 0$$

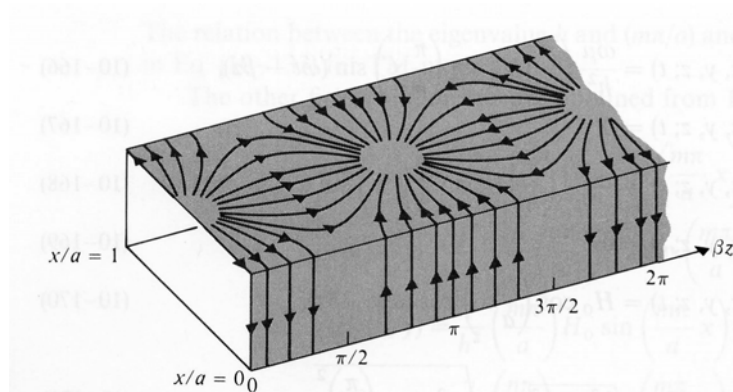
where  $\beta = \sqrt{k^2 - h^2} = \sqrt{\omega^2 \mu\epsilon - (\frac{\pi}{a})^2}$  (b)  $(\frac{dx}{dz})_H = \frac{\beta}{h^2}(\frac{\pi}{a}) \tan(\frac{\pi}{a}x) \tan \beta z$

(c)  $\vec{J}_s = \hat{a}_n \times \vec{H}$ . At  $t=0$ ,  $\vec{J}_s(x=0) = -\hat{y}H_z(0, y, z; 0) = -\hat{y}H_0 \cos \beta z$

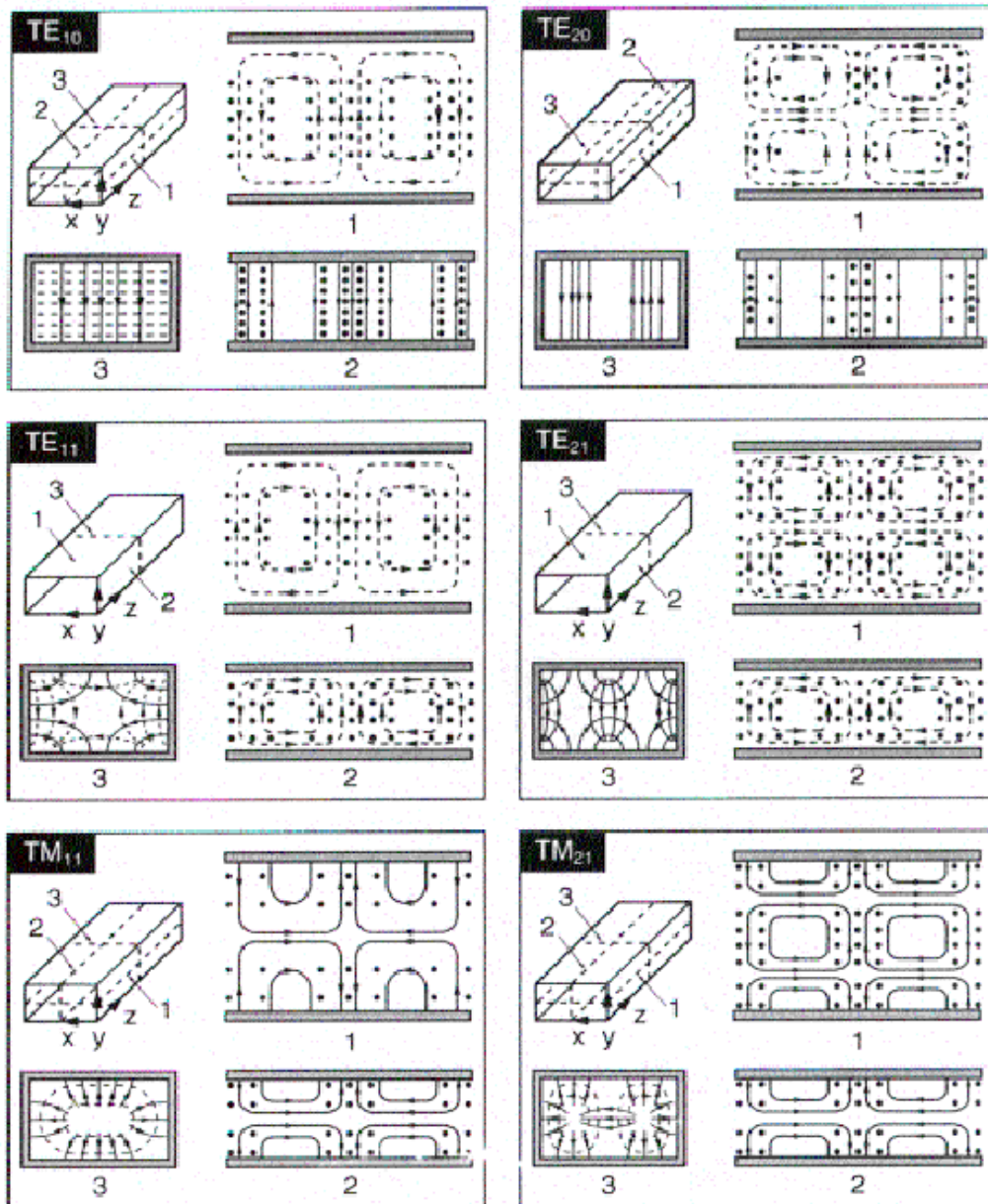
$$\vec{J}_s(x=a) = \hat{y}H_z(a, y, z; 0) = \vec{J}_s(x=0)$$

$$\vec{J}_s(y=0) = \hat{x}H_z(x, 0, z; 0) - \hat{z}H_x(x, 0, z; 0) = \hat{x}H_0 \cos(\frac{\pi}{a}x) \cos(\beta z) - \hat{z} \frac{\beta}{h^2}(\frac{\pi}{a})H_0 \sin(\frac{\pi}{a}x) \sin \beta z$$

$$\vec{J}_s(y=b) = -\vec{J}_s(y=0)$$



**Eg. The eigenmodes in the rectangular waveguide.**



**Modes in Rectangular Waveguide**

**Bending and T-Branched of Waveguides:**



### Attenuation in the rectangular waveguide:

$$\alpha = \alpha_d + \alpha_c, \text{ where } \alpha_d = \frac{\sigma \eta}{2\sqrt{1 - (f_c/f)^2}} \text{ and } \alpha_c = \frac{P_L(z)}{2P(z)}.$$

Consider  $TE_{10}$  mode:

$$P(z) = \int_0^b \int_0^a -\frac{1}{2} (E_y^0)(H_x^0)^* dx dy = \frac{1}{2} \omega \mu \beta \left(\frac{a}{\pi}\right)^2 H_0^2 \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx dy = \omega \mu \beta ab \left(\frac{aH_0}{2\pi}\right)^2$$

and

$$\begin{cases} \bar{J}_s^0(x=0) = \bar{J}_s^0(x=a) = -\hat{y}_y H_z^0(x=0) = -\hat{y}_y H_0 \\ \bar{J}_s^0(y=0) = -\bar{J}_s^0(y=b) = \hat{x} H_z^0(y=0) - \hat{z} H_x^0(y=0) = \hat{x} H_0 \cos\left(\frac{\pi}{a}x\right) - \hat{z} \frac{\beta a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right) \end{cases}$$

$$P_L(z) = 2[P_L(z)]_{x=0} + 2[P_L(z)]_{y=0}, \quad [P_L(z)]_{x=0} = \int_0^b \frac{1}{2} |J_s^0(x=0)|^2 R_s dy = \frac{b}{2} H_0^2 R_s$$

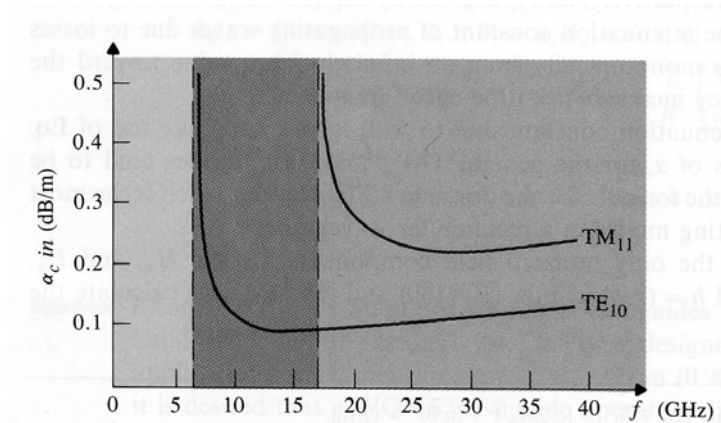
$$\text{and } [P_L(z)]_{y=0} = \int_0^a \frac{1}{2} [|J_{sx}^0(y=0)|^2 + |J_{sy}^0(y=0)|^2] R_s dx = \frac{a}{4} \left[1 + \left(\frac{\beta a}{\pi}\right)^2\right] H_0^2 R_s$$

$$\Rightarrow P_L(z) = \left\{ b + \frac{a}{2} \left[1 + \left(\frac{\beta a}{\pi}\right)^2\right] \right\} H_0^2 R_s = \left[ b + \frac{a}{2} \left(\frac{f_c}{f}\right)^2 \right] H_0^2 R_s$$

$$(\alpha_c)_{TE_{10}} = \frac{R_s [1 + (2b/a)(f_c/f)^2]}{\eta b \sqrt{1 - (f_c/f)^2}} = \frac{1}{\eta b} \sqrt{\frac{\pi f \mu_c}{\sigma_c [1 - (f_c/f)^2]}} \left[1 + \frac{2b}{a} \left(\frac{f_c}{f}\right)^2\right], \quad R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

$$\text{Similar approach } \Rightarrow (\alpha_c)_{TM_{11}} = \frac{2R_s [(b/a^2) + (a/b^2)]}{\eta ab \sqrt{1 - (f_c/f)^2} [(1/a^2) + (1/b^2)]}$$

$$\text{General Cases: } \begin{cases} (\alpha_c)_{TE_{mn}} = \frac{2R_s}{\eta b \sqrt{1 - (f_c/f)^2}} \cdot \left\{ 1 + \left(\frac{b}{a}\right) \left(\frac{f_c}{f}\right)^2 + \left[\frac{1}{2} - \left(\frac{f_c}{f}\right)^2\right] \cdot \frac{\frac{b}{a} \left(\frac{b}{a} m^2 + n^2\right)}{\frac{b^2}{a^2} m^2 + n^2} \right\} \\ (\alpha_c)_{TM_{mn}} = \frac{2R_s}{\eta b \sqrt{1 - (f_c/f)^2}} \cdot \frac{m^2 \left(\frac{b}{a}\right)^3 + n^2}{m^2 \left(\frac{b}{a}\right)^2 + n^2} \end{cases}$$



**Ex. A  $TE_{10}$  wave at 10GHz propagates in a brass  $\sigma_c=1.57 \times 10^7$  (S/m) rectangular waveguide with inner dimensions  $a=1.5\text{cm}$  and  $b=0.6\text{cm}$ , which is filled with  $\epsilon_r=2.25$ ,  $\mu_r=1$ , loss tangent= $4 \times 10^{-4}$ . Determine (a) the phase constant, (b) the guide wavelength, (c) the phase velocity, (d) the wave impedance, (e) the attenuation constant due to loss in the dielectric, and (f) the attenuation constant due to loss in the guide walls.**

$$\text{(Sol.) } f=10^{10}\text{Hz}, \quad \lambda = \frac{v}{f} = \frac{3 \times 10^8}{\sqrt{2.25} \times 10^{10}} = \frac{2 \times 10^8}{10^{10}} = 0.02\text{m}$$

$$\text{For } TE_{10} \text{ mode, } f_c = \frac{v}{2a} = \frac{2 \times 10^8}{2 \times (1.5 \times 10^{-2})} = 0.667 \times 10^{10} \text{ Hz}$$

$$\beta = \frac{\omega}{v} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 234 \text{ rad/m}, \quad \lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} = 0.0268\text{m}$$

$$v_p = \frac{v}{\sqrt{1 - (f_c/f)^2}} = 2.68 \times 10^8 \text{ m/s}, \quad Z_{TE_{10}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 - (f_c/f)^2}} = 337.4(\Omega)$$

$$\sigma = 4 \times 10^{-4} \omega \epsilon = 5 \times 10^{-4} \text{ S/m}, \quad \alpha_d = \frac{\sigma}{2} Z_{TE_{10}} = 0.084 \text{ Np/m} = 0.73 \text{ dB/m}$$

$$R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} = 0.05101(\Omega),$$

$$\alpha_c = \frac{R_s [1 + (2b/a)(f_c/f)^2]}{\eta b \sqrt{1 - (f_c/f)^2}} = 0.0526 \text{ Np/m} = 0.457 \text{ dB/m}.$$

**Ex. (a) Determine the value of  $f_c/f$  at which the attenuation constant due to conductor losses in an  $a \times b$  rectangular waveguide for the  $TE_{10}$  mode is a minimum. What is the minimum obtainable  $\alpha_c$  in a  $2\text{cm} \times 1\text{cm}$  guide? At what frequency? (b) Determine the value of  $(f/f_c)_{TM_{11}}$  at which this attenuation constant is a minimum.**

$$\text{(Sol.) (a) } (\alpha_c)_{TE_{10}} = \frac{1}{\eta b} \cdot \sqrt{\frac{\pi f \mu_c}{\sigma_c [1 - (f_c/f)^2]}} \cdot \left[1 + \frac{2b}{a} \left(\frac{f_c}{f}\right)^2\right], \quad \frac{d(\alpha_c)_{TE_{10}}}{df} = 0$$

$$\Rightarrow f = f_c \cdot \left[ \frac{(6b+3a) - \sqrt{(6b+3a)^2 - 8ab}}{2a} \right]^{-1/2}$$

$$\text{(b) } (\alpha_c)_{TM_{11}} = \frac{2R_s (b/a^2 + a/b^2)}{\eta ab \sqrt{1 - (f_c/f)^2} \cdot (1/a^2 + 1/b^2)}, \quad \frac{d(\alpha_c)_{TM_{11}}}{df} = 0 \Rightarrow f = \sqrt{3} f_c$$



**Eg. An air-filled rectangular waveguide made of copper and having transverse dimensions  $a=7.20\text{cm}$  and  $b=3.40\text{cm}$  operates at a frequency  $3\text{GHz}$  in the dominant mode. Find (a)  $f_c$ , (b)  $\lambda_g$ , (c)  $\alpha_c$ , and (d) the distance over which the field intensities of the propagating wave will be attenuated by 50%.**

$$\text{(Sol.) } \sigma_c = 5.8 \times 10^7 \text{ S/m (a) } (f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}} = 2.083\text{GHz} < 3\text{GHz}$$

$$\text{(b) } \lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} = 0.109\text{m}$$

$$\text{(c) } (\alpha_c)_{TE_{10}} = \frac{1}{\eta b} \sqrt{\frac{\pi f \mu_c}{\sigma_c [1-(f_c/f)^2]}} \cdot \left[1 + \frac{2b}{a} \left(\frac{f_c}{f}\right)^2\right] = 2.26 \times 10^{-3} \text{ Np/m}$$

$$\text{(d) } 0.5 = e^{-\alpha_c d} \Rightarrow d = 307.25\text{m}$$

**Eg. An average power of  $1\text{kW}$  at  $10\text{GHz}$  is to be delivered to an antenna at the  $TE_{10}$  mode by an air-filled rectangular copper waveguide  $1\text{m}$  long and having sides  $a=2.25\text{cm}$  and  $b=1.00\text{cm}$ . Find (a) the attenuation constant due to conductor losses, (b) the maximum values of the electric and magnetic field intensities within the waveguide, (c) the maximum value of the surface current density on the conducting walls, (d) the total amount of average power dissipated in the waveguide. [台大電研]**

$$\text{(Sol.) (a) } (f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}} = 6 \times 10^9 \text{ (Hz)}, \quad f = 10 \times 10^9 \text{ Hz}, \quad \sqrt{1-(f_c/f)^2} = 0.7454$$

$$(\alpha_c)_{TE_{10}} = \frac{1}{\eta b} \cdot \sqrt{\frac{\pi f \mu_c}{\sigma_c [1-(f_c/f)^2]}} \cdot \left[1 + \frac{2b}{a} \left(\frac{f_c}{f}\right)^2\right] = 0.13 \text{ Np/m}$$

$$\text{(b) } P = \omega \mu \beta a b \left(\frac{a H_0}{2\pi}\right)^2, \quad \beta = \omega \sqrt{\mu\epsilon} \cdot \sqrt{1-(f_c/f)^2} = 156.1 \text{ rad/m}, \quad h^2 = \left(\frac{\pi}{a}\right)^2$$

$$\Rightarrow 1000 = 3.56 \times 10^{-2} H_0^2 \Rightarrow H_0 = 167 \text{ A/m} \Rightarrow |E_{\max}| = |E_y^0(x, y)|_{\max} = \frac{\omega \mu}{h^2} \cdot \frac{\pi}{a} H_0 = 94800 \text{ V/m}$$

$$|H_x^0(x, y)|_{\max} = \frac{\beta a}{\pi} H_0 = 187.4 \text{ A/m}, \quad |H_z^0(x, y)|_{\max} = H_0 = 167 \text{ A/m}.$$

If at input end, all factor  $\times e^{0.13}$  (=1.138)

$$\text{(c) } \vec{J}_s(x=0) = -\hat{y} H_0 \cos \beta z \Rightarrow |\vec{J}_s(x=0)| = H_0, \quad \vec{J}_s(y=0) = \hat{y} \times (\hat{x} H_x^0 + \hat{z} H_z^0)_{y=0}$$

$$\Rightarrow |J_s(y=0)| = \frac{E_0}{\eta_0} \cdot \left\{ \left(\frac{f_c}{f}\right)^2 + \left[1 - 2\left(\frac{f_c}{f}\right)^2 \cdot \sin\left(\frac{\pi x}{\sigma}\right)\right] \right\}^{1/2}, \quad \therefore \text{Max of } |J_s(y=0)| = \frac{E_0}{\eta_0} \cdot \sqrt{1-(f_c/f)^2}$$

At the input end, this factor  $\times 1.138$

$$\text{(d) } \Delta P = 1000 \times [e^{2\alpha_c l} - 1] = 26.2 \text{ W} \quad (\because P \propto |E|^2 \propto e^{-2\alpha_c l})$$

**Eg. Find the maximum amount of 10GHz average power that can be transmitted through an air-filled rectangular waveguide  $a=2.25\text{cm}$ ,  $b=1.00\text{cm}$  at the  $TE_{10}$  mode without a breakdown. (Without breakdown:  $|E|_{\text{max}} = 3 \times 10^6 \text{V/m}$ )**

(Sol.)  $H_0 = 5305 \text{A/m}$ ,  $P(z) = \omega\mu\beta ab(aH_0/2\pi)^2 \approx 1 \text{MW}$

#### 6-4 Circular Waveguides



**Circular waveguide's equations:**

$$\begin{cases} E_r = -\frac{j}{h^2} \left[ \beta \frac{\partial E_z}{\partial r} + \frac{\omega\mu}{r} \frac{\partial H_z}{\partial \phi} \right], & H_r = \frac{j}{h^2} \left[ \frac{\omega\varepsilon}{r} \frac{\partial E_z}{\partial \phi} - \beta \frac{\partial H_z}{\partial r} \right] \\ E_\phi = \frac{j}{h^2} \left[ -\frac{\beta}{r} \frac{\partial E_z}{\partial \phi} + \omega\mu \frac{\partial H_z}{\partial r} \right], & H_\phi = -\frac{j}{h^2} \left[ \omega\varepsilon \frac{\partial E_z}{\partial r} + \frac{\beta}{r} \frac{\partial H_z}{\partial \phi} \right] \end{cases}$$

**Case 1  $TM_{np}$  modes:  $H_z=0$ , and  $\nabla_t^2 E_z + h^2 E_z = 0$**

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + h^2 E_z = 0$$

$$E_z^0(r, \phi) = C_n J_n(hr) \cos n\phi \quad \text{and} \quad E_z(r, \phi, z) = E_z^0(r, \phi) e^{-\gamma z}$$

$$\Rightarrow \begin{cases} E_r^0 = -\frac{j\beta}{h} C_n J'_n(hr) \cos n\phi, & E_\phi^0 = \frac{j\beta n}{h^2 r} C_n J_n(hr) \sin n\phi \\ H_r^0 = -\frac{j\omega\varepsilon n}{h^2 r} C_n J_n(hr) \sin n\phi, & H_\phi^0 = -\frac{j\omega\varepsilon}{h} C_n J'_n(hr) \cos n\phi \end{cases}$$

$E_z^0 = H_z^0 = 0$  at  $r=a$  and  $h^2 = \gamma^2 + k^2 \Rightarrow h$  fulfills  $J_n(ha) = 0$  (the first root of  $J_0(x)$  is 2.405)

For  $TM_{01}$  mode, cutoff frequency:  $(f_c)_{TM_{01}} = \frac{h_{TM_{01}}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.383}{a\sqrt{\mu\varepsilon}} \quad (h_{TM_{01}} = \frac{2.405}{a})$

**Note:**  $h$  fulfills the  $p^{\text{th}}$  root of  $J_n(ha) = 0$  for  $TM_{np}$  mode, and  $h$  fulfills the  $p^{\text{th}}$  root of  $J'_n(ha) = 0$  for  $TE_{np}$  mode.

Case 2  $TE_{np}$  modes:  $E_z=0, \nabla_t^2 H_z + h^2 H_z = 0 \Rightarrow H_z^0(r, \phi) = C'_n J_n(hr) \cos n\phi$

Waveguide's equations

$$\Rightarrow \begin{cases} H_r^0 = -\frac{j\beta}{h} C'_n J'_n(hr) \cos n\phi, & H_\phi^0 = \frac{j\beta n}{h^2 r} C'_n J_n(hr) \sin n\phi \\ E_r^0 = \frac{j\omega\mu n}{h^2 r} C'_n J_n(hr) \sin n\phi, & E_\phi^0 = -\frac{j\omega\mu}{h} C'_n J'_n(hr) \cos n\phi \end{cases}$$

$$E_z^0 = 0, \frac{\partial H_z^0}{\partial r} = 0 \text{ at } r=a \Rightarrow h \text{ fulfills } J'_n(ha)=0 \text{ (the first root of } J'_1(x) \text{ is 1.841)}$$

Cutoff frequency of  $TE_{11}$  mode:  $(f_c)_{TE_{11}} = \frac{h_{TE_{11}}}{2\pi\sqrt{\mu\epsilon}} = \frac{0.293}{a\sqrt{\mu\epsilon}} \left( h_{TE_{11}} = \frac{1.841}{a} \right)$

Note:  $TE_{11}$  mode is the fundament (dominant) mode of a circular waveguide.

**Summary of Wave Types for Circular Guides\***

Wave Type	$TM_{01}$	$TM_{02}$	$TM_{11}$	$TE_{11}$	$TE_{21}$
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	$E_z, E_r, H_\phi$	$E_z, E_r, H_\phi$	$E_r, E_\phi, H_z, H_\phi$	$H_z, H_r, E_\phi$	$H_z, H_r, H_\phi, E_r, E_\phi$
$p_{\omega}$ or $p'_{\omega}$	2.405	5.52	3.83	3.83	1.84
$(k_c)\omega$	$\frac{2.405}{a}$	$\frac{5.52}{a}$	$\frac{3.83}{a}$	$\frac{3.83}{a}$	$\frac{1.84}{a}$
$(\alpha_c)\omega$	2.61e	1.14a	1.64a	1.64a	3.41a
$(\beta_c)\omega$	$\frac{0.383}{a\sqrt{\mu\epsilon}}$	$\frac{0.877}{a\sqrt{\mu\epsilon}}$	$\frac{0.609}{a\sqrt{\mu\epsilon}}$	$\frac{0.609}{a\sqrt{\mu\epsilon}}$	$\frac{0.293}{a\sqrt{\mu\epsilon}}$
Attenuation due to imperfect conductors	$\frac{R_s}{\omega\epsilon} \frac{1}{\sqrt{1-(\alpha_c/\beta)^2}}$	$\frac{R_s}{\omega\epsilon} \frac{1}{\sqrt{1-(\alpha_c/\beta)^2}}$	$\frac{R_s}{\omega\epsilon} \frac{1}{\sqrt{1-(\alpha_c/\beta)^2}}$	$\frac{R_s}{\omega\epsilon} \frac{(\alpha_c/\beta)^2}{\sqrt{1-(\alpha_c/\beta)^2}}$	$\frac{R_s}{\omega\epsilon} \frac{1}{\sqrt{1-(\alpha_c/\beta)^2}} \left[ \left(\frac{\alpha_c}{\beta}\right)^2 + 0.420 \right]$

\* Electric field lines are shown solid and magnetic field lines are dashed.

Eg. (a) A 10GHz signal is to be transmitted inside a hollow circular conducting pipe. Determine the inside diameter of the pipe such that its lowest cutoff frequency is 20% below this signal frequency. (b) If the pipe is to operate at 15GHz, what waveguide modes can propagate in the pipe?

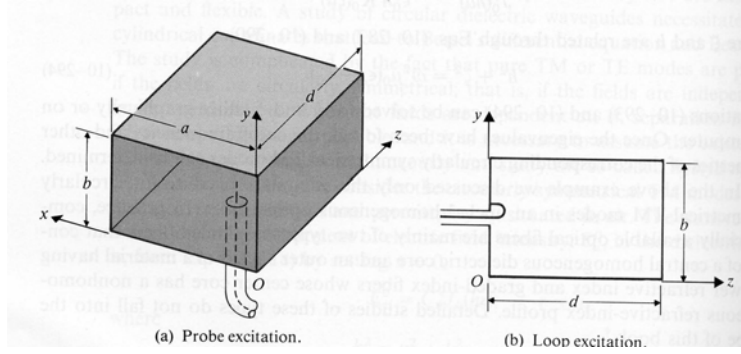
(Sol.) (a)  $(f_c)_{TE_{11}} = \frac{0.293}{a\sqrt{\mu_0\epsilon_0}} = \frac{0.879}{a} \times 10^8 \text{ (Hz)}, 10 \times (1-20\%) = 8, 2a = 0.022m.$

(b)  $f_c$  of waveguide with  $a=0.011(m)$  is  $(f_c)_{TE_{11}} = 8GHz < 15GHz.$

$$(f_c)_{TM_{01}} = \frac{2.405}{2\pi a\sqrt{\mu_0\epsilon_0}} = 10.45(GHz), (f_c)_{TE_{21}} = \frac{3.054}{2\pi a\sqrt{\mu_0\epsilon_0}} = 13.27(GHz)$$

Zeros of $J_n(x), x_{np}$				Zeros of $J'_n(x), x'_{np}$			
$p \backslash n$	$n=0$	$n=1$	$n=2$	$p \backslash n$	$n=0$	$n=1$	$n=2$
1	2.405	3.832	5.136	1	3.832	1.841	3.054
2	5.520	7.016	8.417	2	7.016	5.331	6.706

## 6-5 Rectangular Cavity Resonators



**Case 1  $TM_{mnp}$  mode:**  $H_z=0$ , neither  $m$  nor  $n=0$ ,  $p$  can be 0.

$$\left\{ \begin{array}{l} E_z(x, y, z) = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right) \\ E_x(x, y, z) = -\frac{1}{h^2} \left(\frac{m\pi}{a}\right) \left(\frac{p\pi}{d}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right) \\ E_y(x, y, z) = -\frac{1}{h^2} \left(\frac{n\pi}{b}\right) \left(\frac{p\pi}{d}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right), \\ H_x(x, y, z) = \frac{j\omega\epsilon}{h^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right) \\ H_y(x, y, z) = -\frac{j\omega\epsilon}{h^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right) \end{array} \right.$$

where  $h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$

**Case 2  $TE_{mnp}$  mode:**  $E_z=0$ ,  $p \neq 0$ . Either  $m$  or  $n=0$ , but not both.

$$\left\{ \begin{array}{l} H_z(x, y, z) = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right) \\ E_x(x, y, z) = \frac{j\omega\mu}{h^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right) \\ E_y(x, y, z) = -\frac{j\omega\mu}{h^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right) \\ H_x(x, y, z) = -\frac{1}{h^2} \left(\frac{m\pi}{a}\right) \left(\frac{p\pi}{d}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right) \\ H_y(x, y, z) = -\frac{1}{h^2} \left(\frac{n\pi}{b}\right) \left(\frac{p\pi}{d}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right) \end{array} \right.$$

Both have the same resonant frequency (degenerate modes):  $f_{mnp} = \frac{1}{2\sqrt{\mu\epsilon}} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$

**Note:**  $TE_{101}$  mode is the dominant mode of the rectangular resonator in case of  $a > b < d$ .

**Eg. Given an air-filled lossless rectangular cavity resonator with dimensions  $8\text{cm} \times 6\text{cm} \times 5\text{cm}$ , find the first twelve lowest-order modes and their resonant frequencies.**

$$\text{(Sol.) } f_r = \frac{c}{2} \sqrt{\left(\frac{m}{0.08}\right)^2 + \left(\frac{n}{0.06}\right)^2 + \left(\frac{p}{0.05}\right)^2} = 1.5 \times 10^{10} \cdot \sqrt{\left(\frac{m}{8}\right)^2 + \left(\frac{n}{6}\right)^2 + \left(\frac{p}{5}\right)^2}$$

$$TM_{110} \Rightarrow f_r = 3.125\text{GHz}, \quad TE_{101} \Rightarrow f_r = 3.54\text{GHz},$$

$$TE_{111}, TM_{111} \Rightarrow f_r = 4.33\text{GHz}, \dots\dots$$

**Quality Factor of Rectangular Cavity Resonators:**  $Q = \frac{\omega W}{P_L}$ ,  $W = W_e + W_m$

$Q = 2\pi \times (\text{Time-average energy stored at a resonant frequency}) / (\text{Energy dissipated in one period})$

Consider  $TE_{101}$  mode:

$$W_e = \frac{\epsilon_0}{4} \int |E_y|^2 dv = \frac{\epsilon_0 \omega^2 \mu_0^2 \pi^2}{4h^4 a^2} H_0^2 \int_0^d \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right) dx dy dz = \frac{\epsilon_0 \omega_{101}^2 \mu_0^2 \pi^2}{4\pi^2} H_0^2 \left(\frac{a}{2}\right) b \left(\frac{d}{2}\right)$$

$$= \frac{1}{4} \epsilon_0 \mu_0^2 a^3 b d f_{101}^2 H_0^2$$

$$W_m = \frac{\mu_0}{4} \int \{|H_x|^2 + |H_y|^2\} dv = \frac{\mu_0}{4} H_0^2 \int_0^d \int_0^b \int_0^a \left\{ \frac{\pi^4}{h^4 a^2 d^2} \sin^2\left(\frac{\pi}{a}x\right) \cos^2\left(\frac{\pi}{d}z\right) + \cos^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right) \right\} dx dy dz$$

$$= \frac{\mu}{4} H_0^2 \left\{ \frac{a^2}{d^2} \left(\frac{a}{2}\right) b \left(\frac{d}{2}\right) + \left(\frac{a}{2}\right) b \left(\frac{d}{2}\right) \right\} = \frac{\mu_0}{16} abd \left(\frac{a^2}{d^2} + 1\right) H_0^2$$

where  $h^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2$  and the resonant frequency:  $(f_r)_{TE_{101}} = \frac{1}{2\sqrt{\mu_0 \epsilon_0}} \cdot \sqrt{\frac{1}{a^2} + \frac{1}{d^2}}$

At resonance,  $W = 2W_e = 2W_m = \frac{\mu_0}{8} \cdot H_0^2 abd \left(\frac{a^2}{d^2} + 1\right)$

Power loss per unit area:  $P_{av} = \frac{1}{2} |J_s|^2 R_s = \frac{1}{2} |H|^2 R_s$

$$P_L = \oint P_{av} ds = R_s \left\{ \int_0^b \int_0^a |H_x(z=0)|^2 dx dy + \int_0^d \int_0^b |H_z(x=0)|^2 dy dz + \int_0^d \int_0^a |H_x|^2 dx dz + \int_0^d \int_0^b |H_z|^2 dx dz \right\}$$

$$= \frac{R_s H_0^2}{2} \left\{ \frac{a^2}{d} \left(\frac{b}{d} + \frac{1}{2}\right) + d \left(\frac{b}{a} + \frac{1}{2}\right) \right\} \Rightarrow Q_{TE_{101}} = \frac{\pi f_{101} \mu_0 abd (a^2 + d^2)}{R_s [2b(a^3 + d^3) + ad(a^2 + d^2)]}$$

Similarly, the expression for the  $Q$  of an air-filled  $a \times b \times d$  rectangular resonator for the

$TM_{110}$  mode is  $(Q)_{TM_{110}} = \frac{\pi(f_r)_{TM_{110}} \mu_0 abd (a^2 + b^2)}{R_s [2d(a^3 + d^3) + ab(a^2 + b^2)]}$ .

**Eg. An air-filled rectangular cavity with brass walls  $\sigma=1.57 \times 10^7 (S/m)$  has the following dimensions:  $a=4cm$ ,  $b=3cm$ , and  $d=5cm$ . (a) Determine the dominant mode and its resonant frequency for this cavity. (b) Find the  $Q$  and the time-average stored electric and magnetic energies at the resonant frequency, assuming  $H_0$  to be  $0.1A/m$ .**

(Sol.) (a)  $f_r = \frac{c}{2} \sqrt{\left(\frac{m}{0.04}\right)^2 + \left(\frac{n}{0.03}\right)^2 + \left(\frac{p}{0.05}\right)^2}$ , dominant mode:  $TE_{101}$ ,  $(f_r)_{TE_{101}} = 4.8GHz$

(b)  $(Q)_{TE_{101}} = \frac{\pi(f_r)_{TE_{101}} \mu_0 abd(a^2 + d^2)}{R_s [2b(a^3 + b^3) + ad(a^2 + d^2)]} = 6869$ ,  $R_s = \sqrt{\frac{\pi(f_r)_{TE_{101}} \mu_0}{\sigma_c}}$

At the resonant frequency,  $W_e = W_m = \frac{\epsilon_0}{4} \mu^2 a^3 bd (f_r)_{TE_{101}}^2 H_0^2 = 7.73 \times 10^{-14} (J)$

**Eg. For an air-filled rectangular copper cavity resonator, determine how much  $b$  should be increased in order to make  $Q$  20% higher.**

(Sol.)  $\frac{Q_2}{Q_1} = 1.2 = \sqrt{\frac{b_2}{b_1}} \Rightarrow b_2 = 1.44b_1$

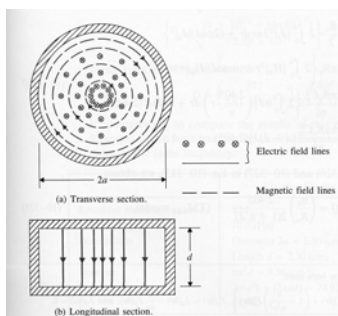
**Eg. (a) What should be the size of a hollow cubic cavity made of copper in order for it to have a dominant resonant frequency of  $10GHz$ ? (b) Find the  $Q$  at that frequency.**

(Sol.) (a) For a cubic cavity,  $a=b=d$ ,  $TM_{110}$ ,  $TE_{011}$ , and  $TE_{101}$  are degenerate dominant modes.  $f_{101} = \frac{3 \times 10^8}{\sqrt{2}a} = 10^{10} (Hz)$ ,  $a = \frac{3 \times 10^8}{\sqrt{2} \times 10^{10}} = 2.12 \times 10^{-2} m$ .

(b)  $Q_{101} = \frac{\pi f_{101} \mu_0 a}{3R_s} = \frac{a}{3} \sqrt{\pi f_{101} \mu_0 \sigma}$

For copper,  $\sigma = 5.80 \times 10^7 (S/m)$ ,  $Q_{101} = \left(\frac{2.12}{3} \times 10^{-2}\right) \sqrt{\pi 10^{10} (4\pi 10^{-7}) (5.80 \times 10^7)} = 10700$

### 6-6 Circular Cavity Resonators



For an air-filled circular cylindrical cavity resonator of radius  $a$  and length  $d$ . The resonant frequencies are

$$(f_r)_{TM_{mnp}} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{X_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}, \text{ where } J_m(X_{mn})=0$$

$$(f_r)_{TE_{mnp}} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{X'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}, \text{ where } J'_m(X'_{mn})=0$$

In case of  $2d > 2a > d$ , the dominant mode of the circular cylindrical cavity is  $TM_{010}$  mode:

$$E_z = C_0 J_0(hr) = C_0 J_0\left(\frac{2.405}{a}r\right), \quad H_\phi = -\frac{jC_0}{\eta_0} J'_0(hr) = \frac{jC_0}{\eta_0} J_1\left(\frac{2.405}{a}r\right)$$

**Quality factor of the  $TM_{010}$  mode:**  $Q_{TM_{010}} = \frac{\omega W}{P_L}$ .

$$W = 2W_e = \frac{\epsilon_0}{2} \int_v |E_z|^2 dv = \frac{\epsilon_0 C_0^2}{2} (2\pi d) \int_0^a J_0^2\left(\frac{2.405}{a}r\right) r dr = (\pi\epsilon_0 d) C_0^2 \left[\frac{a}{2} J_1^2(2.405)\right]$$

$$P_L = \frac{R_s}{2} \left\{ 2 \int_0^a |J_r|^2 2\pi r dr + (2\pi a d) |J_z|^2 \right\} = \pi R_s \left\{ 2 \int_0^a |H_\phi|^2 r dr + (ad) |H_\phi(r=a)|^2 \right\}$$

$$= \frac{\pi R_s C_0^2}{\eta_0^2} \left\{ 2 \int_0^a J_1^2\left(\frac{2.405}{a}r\right) dr + (ad) J_1^2(2.405) \right\} = \frac{\pi a R_s C_0^2}{\eta_0^2} (a+d) J_1^2(2.405)$$

$$\Rightarrow Q_{TM_{010}} = \left(\frac{\eta_0}{R_s}\right) \frac{2.405}{2(1+a/d)}, \quad (f_r)_{TM_{010}} = \frac{2.405}{2\pi a \sqrt{\mu_0 \epsilon_0}} = \frac{0.115}{a} \times 10^9 \text{ Hz}$$

**Ex. A hollow circular cylindrical cavity resonator is to be constructed of copper such that its length  $d$  equals its diameter  $2a$ . (a) Determine  $a$  and  $d$  for a resonant frequency of 10GHz at the  $TM_{010}$  mode. (b) Find the  $Q$  of the cavity at resonance.**

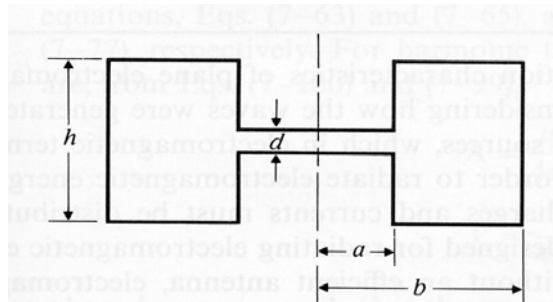
(Sol.) (a)  $(f_r)_{TM_{010}} = \frac{0.115}{a} \times 10^9 = 10 \times 10^9$ ,  $a = 1.15 \times 10^{-2} \text{ m}$ ,  $d = 2a = 2.30 \text{ cm}$ .

(b)  $R_s = \sqrt{\frac{\pi \mu_0}{\sigma}} = \sqrt{\frac{\pi \times 10^{-7} \times (4\pi \times 10^{-7})}{5.80 \times 10^7}} = 2.61 \times 10^{-2} (\Omega)$ ,  $Q = \left(\frac{120\pi}{2.61 \times 10^{-2}}\right) \frac{2.405}{2(1+1/2)} = 11,580$ .

**Ex. In some microwave applications, ring-shaped cavity resonators with a very narrow center part are used. A cross section of such a resonator is shown in the figure, in which  $d$  is very small in comparison with the resonant wavelength. Assuming that this the narrow center part and the inductance of the rest of the structure, find (a) the approximate resonant frequency. (b) the approximate resonant wavelength.**

(Sol.)  $C = \frac{\epsilon \pi a^2}{d}$ ,  $L = \frac{\mu h}{2\pi} \ln\left(\frac{b}{a}\right)$

(a)  $f_r = \frac{1}{2\pi \sqrt{LC}}$  (b)  $\lambda_r = \frac{1}{f_r \sqrt{\mu \epsilon}}$



## 6-7 Excitations of Waveguides

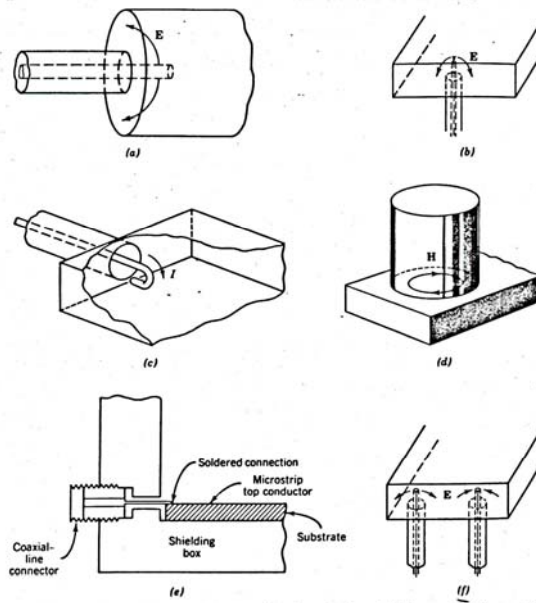
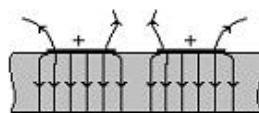
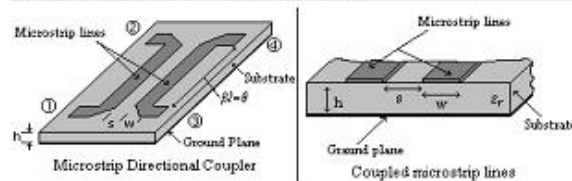
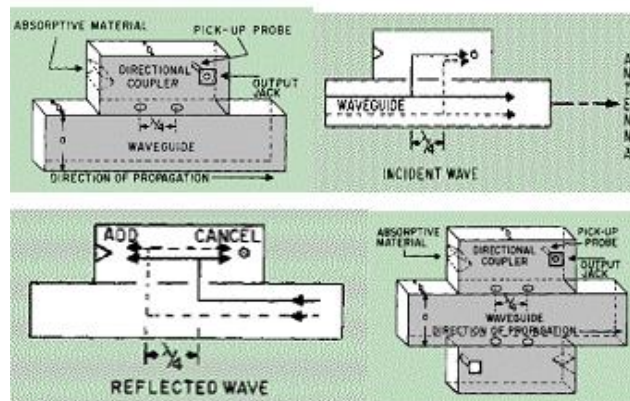
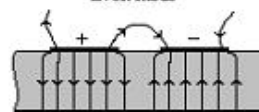


Fig. 6-7. (a) Antenna in end of circular guide for excitation of  $TM_{01}$  wave. (b) Antenna in bottom of rectangular guide for excitation of the  $TE_{10}$  wave. (c) Loop in end of rectangular guide for excitation of  $TE_{10}$  wave. (d) Junction between circular guide ( $TM_{01}$  wave) and rectangular guide ( $TE_{10}$  wave); large-aperture coupling. (e) Coaxial line coupling to microstrip. (f) Excitation of the  $TE_{20}$  wave in rectangular guide by two oppositely phased antennas.

## 6-8 Directional couplers



Even mode



Odd mode

The electric fields for parallel-coupled microstrip lines